Containment of a pair of rotating objects within a container of minimal area or perimeter

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Abstract
Cutting and packing problems arise in many fields of applications and theory. When dealing with irregular objects, an important subproblem is the identification of the optimal clustering of two objects. Within this paper we consider the case, where two irregular one-connected objects whose frontier formed by circular and/or line segments and which can be free rotated, should be placed such that the enclosing rectangle or circle has minimal area, or alternatively, has minimal perimeter. We propose a solution strategy which is based on the concept of phi-functions and provide some examples. Moreover, for the sake of completeness, we give a comprehensive collection of basic phi-functions.

1 Introduction
Cutting and packing problems, also called placement or allocation problems, are of high interest in application as well as in theory. It is well known that even the one-dimensional version of optimal usage of a given resource, the classical knapsack problem, belongs to the class of NP-hard optimization problems. For that reason, most of work related to cutting and packing problems is dealing with heuristic approaches. Nevertheless, the development of exact solution methods is an important task to broaden the range of optimal solvable cases.

In this respect, the investigation of optimization problems where so-called regular objects (meaning rectangles or parallelepipeds/boxes) have to be packed, is considered in a higher percentage of the literature. One reason for that can be the fact that stronger optimality criterion or bounds are available in contrast to cases where irregular objects (already for circles) have to be considered.

One of the successful applicable approaches in case of irregular objects, which leads in general to multi-extremal non-linear optimization problems, is the concept of phi-functions proposed by Stoyan [64] which allows the computation of local optima, and in some cases, even of global optimal solutions.

In this paper, we are going to apply the concept of phi-functions to the counterpart of maximum yield optimization, namely to the computation of minimal material usage needed to obtain all desired objects. In detail, we will investigate the following two-dimensional problem: Let be given two (irregular one-connected) objects whose frontier formed by circular and/or straight line segments. Find a containing region (rectangle or circle) of minimal area or alternatively, of minimal perimeter such that the two objects can be placed completely inside
the containing region without overlapping. We assume that the orientation of the objects is arbitrarily, that means, free rotations of the objects are permitted.

The paper is organized as follows. In the next section, we try to give an overview on related work. Then, for completeness, we describe the concept of phi-functions and apply it to the problem under consideration. We will consider convex and non-convex polygons, circles and objects whose frontier is a combination of straight-line and circular-arc segments (such objects are given in Appendix A together with corresponding phi-functions for pairs of it). The general solution strategy is given together with some examples, and is followed by an outlook. Due to the length of the paper, the extension of the approach to the case of more than two objects is left for a forthcoming paper which will also contain results of numerical experiments of real life problems.

2 Related work

Note, in the following overview only some of the huge number of related articles are mentioned in order to find a balance between the wide field of related areas and length of this section.

Solution approaches to irregular nesting problems are analysed by Dowsland and Dowsland [31]. A tutorial covering the core geometric methodologies currently applied by researchers in cutting and packing of irregular shapes is presented by Bennell and Oliveira [15]. Tools of mathematical modeling of arbitrary object packing problems are given by Bennell, Scheithauer, Stoyan and Romanova [14].

Complexity investigations for cutting and packing problems can be found for instance by Chazelle, Edelsbrunner and Guibas [20], Blazewicz, Drozdowski, Soniewicki and Walkowiak [16], Li and Milenkovic [47], and Chlebk and Chlebkov [24].

Among the plurality of two-dimensional cutting and packing problems the so-called strip packing problem (SPP) is of high interest. In the SPP a given set of objects has to be placed feasibly within a strip of given (fixed) width $W$ while minimizing the height $H$ needed, i.e. a rectangle of minimal area is looked for. (A packing pattern is feasible if all objects are contained completely within the rectangle $W \times H$ and do not overlap each other.) In this case, the objective to minimize the area is equivalent to minimizing the perimeter. This does not remain true if the width $W$ is also variable so that the objective becomes non-linear when the total area has to be minimized.

For the SPP many variants are considered. In case of rectangular objects (rectangles) two main problem classes occur. In the first, the cutting or packing pattern has to possess the so-called guillotine-property, i.e. all objects can be obtained by a sequence of (orthogonal) guillotine cuts. Very well-known heuristics are Next-Fit, First-Fit and Best-Fit (cf. Coffman et al. [25], Baker and Schwarz [10], Imahori and Yagiura [43]) applied in offline situations as well as for online cases. If the guillotine-property is not demanded then Bottom up, Left justified-type heuristics are frequently applied (Baker, Coffman and Rivest [8]). In the latter case, exact methods are proposed based on ILP (integer linear programming) formulations (Padberg [61], Belov and Scheithauer [12]) or a graph-theoretical model (Fekete and Schepers [33]).

The problem of packing circles of various radii is addressed in George, George and Lamar [34], Stoyan and Yaskov [66]. The packing of identical or non-identical circles within a containing circle with minimal radius is investigated in Graham [35] and Wang et al. [69],
ZHANG and DENG [70], HUANG et al. [40], ADDIS, LOCATELLI and SCHOEN [1], respectively. The more general case of circle packing in which the containing region can have different shapes (circle, triangle, square, etc.) is considered in BANSAL and SVIRIDENKO[13].

There exists also work concerning three-dimensional objects, e.g. in EGEBLAD, NIELSEN and ODGAARD [32], BORTFELDT and MACK [17]. In STOYAN, YASKOV and SCHEITHAUER [67] a phi-function based approach is proposed for packing solid spheres into a strip.

Minimum perimeter rectangles that enclose congruent non-overlapping circles are computed by LUBACHEVSKY and GRAHAM [48]. Maximizing the number of identical circles packable within a square, rectangle or circle is also frequently investigated (cf. e.g. SZAJO et al. [68], GRAHAM et al. [35], CUI [27]).

Another broad field in SPP is concerned to the placement of polygonal (convex or non-convex) objects on a (two-dimensional) strip of fixed width and minimal height (or length). Natural applications can be found, for instance, in textile or steel industry. Exemplified we refer to HAISTERMANN and LENGAUER [39], OLIVEIRA and FERREIRA [59], EGEBLAD, NIELSEN and ODGAARD [32], BURKE et al. [18] where various solution approaches are presented. Meta-heuristics like Simulated Annealing (MARQUES, BISPO and SENTIEIRO [49], DOWSLAND [30]) and Genetic Algorithms (MARTENS [50]) (and other) find also application to compute good (but in general not optimal) solutions of such packing problems.

Packing problems with irregular objects, whose frontier can be described by a sequence of line- and arc-segments, are tackled by BURKE et al. [19] using the line and arc no-fit polygon. That paper extends the orbital sliding method of calculating no-fit polygons to enable it to handle arcs and then shows the resultant no-fit polygons being utilised successfully on the two-dimensional irregular packing problem.

Several publications are addressed to containment problems. One type of such problems is as follows: Does a set of given objects (or a single object) fit feasibly within a given containing region? Sometimes rotation of objects is permitted, sometimes not.

The single polygon-containment problem with translations only has been studied in BAKER, FORTUNE and MAHANEY [9]. Rotation of the polygon is additionally allowed in MARTIN and STEPHENSON [51], AVNAI and BOISSONNAT [5], GRINDE and CAVALIER [37]. MILENKOVIC [57]. Two- and three-polygon problems are considered in AVNAI and BOISSONNAT [6], GRINDE and CAVALIER [38].

In MILENKOVIC [54], MILENKOVIC and DANIELS [56], MILENKOVIC [55] the translational containment of multiple polygons within another polygon is investigated, whereas the more general case, allowing also rotations, is considered in MILENKOVIC [58]. In MILENKOVIC and SACKS [53] the authors offer an implementation of Minkowski sum for objects bounded by line segments and circular arcs. GRINDE and CAVALIER [37] use a mathematical programming formulation/model to solve the following containment problem: Given a convex polygon \( Q \) and a (not necessary convex) polygon \( P \), can \( P \) be translated and/or rotated such that \( P \) fits within \( Q \)? This method is used in an algorithm to place small (polygonal) items into (polygonal) holes obtained after placing larger items.

The minimal-area convex enclosure problem consists in finding the relative position of two simple polygons to each other such that the area of their convex enclosure is minimized. It is investigated in GRINDE and CAVALIER [36]. The technique used searches along the envelope (or no-fit polygon). DORI and BEN-BASSAT [29] consider the problem of circumscribing a convex polygon by a polygon of fewer sides with minimal area addition.
The best approximation of a convex figure \( C \) by a pair of homothetic rectangles (i.e. they are similar and parallel), \( r \) and \( R \) with \( r \subseteq C \subseteq R \), is studied by Schwarzkopf, Fuchs, Rote and Welzl [63]. Approximations (inscribing and enclosing) for planar convex sets using axially symmetric polygons are studied by Ahn et al. [3]. Finding the largest area axis-parallel rectangle in a polygon is considered by Daniels, Milekovic and Roth [28]. The computation of the minimum bounding box for a given object (and other problems) is considered by Martin and Stephenson [51].

The problem of enclosing a set of objects by two minimum area rectangles is addressed by Becker et al. [11]. The enclosure of \( n \) points in the plane by two non-convex shapes, rectilinear convex hull and L-shape, having minimal area, is investigated by Bae et al. [7]. Covering (partition) a rectilinear polygon by (into) a minimal number of axis-parallel rectangles are considered by Anil Kumar and Ramesh [4] and Cheng and Lin [21]. A minimum-area rectangle enclosing the projection of a higher-dimensional set (a polytope) is constructed by Kuno [46].

The decision problem, given a set of points in the plane and a set of rectangles (which can be rotated), is there a placement of the rectangles such that all points are covered, is studied by Huang and Wang [41]. A similar problem is addressed by Ahn et al. [2].

Hche and Liebling [42] investigates the computation of minimum area simple (not-necessary convex) pentagons whose vertexes are elements of a given point set. Melissen and Schuur [52] computes the minimal radius needed when a rectangle is covered by six or seven circles. An optimal algorithm for finding minimal area triangles enclosing a given convex polygon is proposed by O’Rourke, Aggarwal, Maddila and Baldwin [60].

Paper Kallrath [44] considers a placement of circles and polygons within rectangular area of minimal area.

The smallest square in which all rectangles of size \( 1/n \times 1/(n+1) \), \( n = 1, 2, \ldots \), can be packed, and the rectangle with smallest area in which all squares of size \( 1/n \times 1/n \), \( n = 3, 4, 5, \ldots \) can be packed are presented by Jennings [45].

Heuristic approaches to large-scale periodic packing of irregular shaped objects on a rectangular sheet are also of interest (cf. Costa, Gomes and Oliveira [26]).

Most of the approaches dealing with the interaction of two (or only few) objects (containment or covering) are used in placement algorithms for larger instances as local decision rules. In this paper, we use a uniform description, the concept of phi-functions, to compute (approximately) a minimum area or minimum perimeter rectangle, or a circle of minimum area, containing two objects which can be translated and/or rotated. (The proposed approach can easily be extended to a higher number of objects.) As objects we consider rectangles, polygons, circles and objects whose frontier is formed by straight-line and circular-arc segments.

3 The concept of phi-functions

Two-dimensional cutting and packing problems with irregular shaped objects occur in footwear, textile, metal cutting, etc. In order to model such optimization problems the interaction of two objects with respect to non-overlapping, intersection or contact has to be
described in an appropriate manner. It has been proved that the usage of so-called phi-functions for pairs of geometric objects is very helpful in this respect, cf. Chernov, Stoyan and Romanova [23], Stoyan and Chugay [62], Stoyan and Yaskov [65].

In any case, we assume that any object $T$ considered here, is a model of a real two-dimensional object, i.e. $T$ is phi-object. Canonically closed point set $T \subset \mathbb{R}^2$ ($T = \text{cl}^i(T) = \text{cl}(\text{int}(T))$ having the same homotopic type as its interior is called phi-object. In particular, any phi-object is called a phi-polygon if its frontier is shaped by means of straight lines, rays or line segments Bennell, Scheithauer, Stoyan and Romanova [14]. To handle arbitrary geometric objects we present them by finite unions of so-called basic objects. In Chernov, Stoyan and Romanova [23] is proved that each phi-object bounded by line segments and circular arcs may be decomposed into basic phi-objects: $T = \bigcup_{i=1}^{n} T_i$, with $T_i \in \mathfrak{T}$, where $\mathfrak{T}$ is the set of basic objects. For details we refer the reader to Chernov, Stoyan and Romanova [23].

The translation of object $T \subset \mathbb{R}^2$ by vector $u = (u_x, u_y) \in \mathbb{R}^2$ and rotation of it (with respect to its reference point) by angle $\theta \in [0,2\pi)$ is defined by

$$T(u, \theta) = \{ \tilde{t} : \tilde{t} = u + D(\theta)t \quad \forall t \in T(0,0) \}$$

where $T(0,0)$ denotes the non-translated and non-rotated object $T$, and where $D(\theta)$ is given by

$$D(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \text{and} \quad D^{-1}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ 

Note, $D(\theta)$ defines clock-wise rotation. For analytical descriptions of relations between a pair of objects $T_1(u^1, \theta_1)$ and $T_2(u^2, \theta_2)$ we employ phi-function technique. Phi-functions allow us to distinguish the following three cases: $T_1(u^1, \theta_1)$ and $T_2(u^2, \theta_2)$ are intersecting so that $T_1(u^1, \theta_1)$ and $T_2(u^2, \theta_2)$ have common interior points; $T_1(u^1, \theta_1)$ and $T_2(u^2, \theta_2)$ do not intersect, i.e. $T_1(u^1, \theta_1)$ and $T_2(u^2, \theta_2)$ do not have common points; $T_1(u^1, \theta_1)$ and $T_2(u^2, \theta_2)$ are in contact, i.e. $T_1(u^1, \theta_1)$ and $T_2(u^2, \theta_2)$ have only common frontier points.

**Definition** Any everywhere defined continuous function $\Phi : \mathbb{R}^6 \to \mathbb{R}$ is called phi-function of $T_1$ and $T_2$ if it possesses the following characteristic properties:

$$\Phi(u^1, \theta_1, u^2, \theta_2) = \begin{cases} > 0, & \text{if } T_1(u^1, \theta_1) \cap T_2(u^2, \theta_2) = \emptyset \\ = 0, & \text{if } \text{int}T_1(u^1, \theta_1) \cap \text{int}T_2(u^2, \theta_2) = \emptyset \\ < 0, & \text{if } \text{fr}T_1(u^1, \theta_1) \cap \text{fr}T_2(u^2, \theta_2) \neq \emptyset. \end{cases}$$

Definition and basic features of phi-functions one may find in Bennell, Scheithauer, Stoyan and Romanova [14], Chernov, Stoyan and Romanova [23].

Now let

$$A = \bigcup_{i=1}^{n_A} A_i, \quad B = \bigcup_{j=1}^{n_B} B_j, \quad \text{and } A_i, B_j \in \mathfrak{T}. \quad (1)$$

A phi-function to characterise the non-overlapping of the pair of objects $A$ and $B$ bounded by line segments and circular arcs has the form

$$\Phi_{AB} = \min\{ \Phi_{ij}, i = 1, 2, ..., n_A, j = 1, 2, ..., n_B \},$$
where $\Phi_{ij}$ denotes a basic phi-function for the pair of objects $A_i$ and $B_j$ with and $A_i$, $B_j \in \mathcal{X}$. The complete class of basic phi-functions is given in Chernov, Stoyan, Romanova and Pankratov [22]. We provide the phi-functions in Appendix A.

4 Mathematical model of the containment problem

In the problem under consideration, we are looking for an axis-parallel rectangle

$$R = R(a, b) = \{(x, y) : 0 \leq x \leq a, \ 0 \leq y \leq b\}$$

of size $a \times b$ in fixed position, or alternatively, for a containing circle

$$C = C(r) = \{(x, y) : x^2 + y^2 \leq r\}$$

with origin as center point. In the following, let $C_0(p)$ denote the desired containing region, i.e. $C_0(p)$ represents either the rectangle $R(a, b)$, or the circle $C(r)$, and $p$ denotes the vector of metrical characteristics belonging to the containing region (either $p = (a, b)$ or $p = (r)$). The approach proposed below can be applied also for other types of containing regions. Moreover, the number of objects to be placed is not restricted to two.

Let $F_k = F_k(p), k = 1, 2, 3$ denote objective functions:

$$F_1(a, b) = ab, \quad F_2(a, b) = a + b, \quad F_3(r) = r.$$

The objective is either to minimize the area, or alternatively, the perimeter, under the condition that two objects, named $A$ and $B$, can be placed completely within $C_0(p)$ without overlap. With other words, we are also looking for translation vectors $u^A$ and $u^B$ and angles $\theta_A$ and $\theta_B$ such that

$$A(u^A, \theta_A) \subset C_0(p), \quad B(u^B, \theta_B) \subset C_0(p), \quad A(u^A, \theta_A) \cap \text{int}B(u^B, \theta_B) = \emptyset.$$

The latter condition is fulfilled if and only if $\Phi_{A,B}(u^A, \theta_A, u^B, \theta_B) \geq 0$. The remaining two conditions can also be modelled using phi-functions if instead of $C_0(p)$ the closure of its complement, $C_0^*(p) = \text{cl}(\mathbb{R}^2 \setminus C(p))$, is used. Let $\Phi_A(u^A, \theta_A, a, b)$ denote the phi-function for the pair of objects $A$ and $C_0^*$. Analogously, let $\Phi_B(u^B, \theta_B, a, b)$ denote the phi-function for the pair of objects $B$ and $C_0^*$. Consequently, we formulate the following problem:

Compute translation vectors $u^A, u^B \in \mathbb{R}^2$, rotation angles $\theta_A, \theta_B \in [0, 2\pi)$, and size parameters $p$ such that $F_k(p) \ (k \in \{1, 2, 3\})$ attains its minimum and

$$\Phi_{A,B}(u^A, \theta_A, u^B, \theta_B) \geq 0, \quad \Phi_A(u^A, \theta_A, p) \geq 0, \quad \Phi_B(u^B, \theta_B, p) \geq 0.$$

Because of a symmetry argument, in case of a rectangular containing region, we may restrict $\theta_A$ to be in $[0, \pi)$. In case of a circular containing region $\theta_A = 0$ is possible. Let $W \subset \mathbb{R}^8$ denote the corresponding set of feasible solutions (the solution space). Then, we can define the following optimisation problem:

$$F_k(u^*) = \min\{F_k(u) : u \in W\}, \quad (k \in \{1, 2, 3\}),$$

where $u = (u^A, \theta_A, u^B, \theta_B, p)$. 

\[ F_k(u^*) = \min\{F_k(u) : u \in W\}, \quad (k \in \{1, 2, 3\}), \]
It is obvious, \( W \neq \emptyset \), and since \( F_k \) is bounded from below, the problem is always solvable. But the problem is difficult to solve for several reasons. The objective \( F_1 \) is non-linear (quadratic) as well as the phi-functions which are composed by min- and max-combinations of affine-linear and non-linear functions including sin- and cos-terms (cf. Chernov, Stoyan and Romanova [23]). The set \( W \) of feasible solutions is non-convex, leading to many local extrema. Each object, we consider here, can be represented in the form

\[
T = \{ (x, y, \theta) : \min_{j=1, \ldots, n} f_j(x, y, \theta) \leq 0 \}
\]

with \( f_j(x, y, \theta) = \max_{k=1, \ldots, n_j} f_{jk}(x, y, \theta) \), \( j = 1, \ldots, n \), where \( f_{jk} \) are radical free affine-linear and non-linear functions including sin- and cos-terms. It follows from definition of \( T \) as a union of basic objects \( T_j, j = 1, \ldots, n \), where

\[
T_j = \{ (x, y, \theta) : f_j(x, y, \theta) \leq 0 \}
\]

Example 1

We consider two arbitrary shaped phi-objects, \( A \) and \( B \), whose frontier is formed by line segments and circular arcs. Each object is given by an ordered collection of frontier elements: \( l_1, l_2, \ldots, l_n \), where \( l_i \in \{ -1, 0, 1 \} \); \( l_i = 1 \) corresponds to a convex arc, \( l_i = 0 \) corresponds to a line segment, and \( l_i = -1 \) corresponds to a concave arc. Each arc is given by a tuple \( g_i = (x_1, y_1, x_2, y_2, r) \), where \( (x_1, y_1) \) and \( (x_2, y_2) \) are coordinates of the end points, \( (x_c, y_c) \) is the center point of the circular arc and \( r \) is the radius of it in the eigen coordinate system of object \( T \). Each straight-line segment is given by a tuple \( g_i = (x_1, y_1, x_2, y_2) \), where \( (x_1, y_1) \) and \( (x_2, y_2) \) are the end points of it in the eigen coordinate system of object \( T \). The input data of two objects \( A \) and \( B \) (Fig. 1) are given the following table.

<table>
<thead>
<tr>
<th>code</th>
<th>( (x_1, y_1) )</th>
<th>( (x_c, y_c) )</th>
<th>( (x_2, y_2) )</th>
<th>( r )</th>
</tr>
</thead>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>(9.517820, 4.368199)</td>
<td>21</td>
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<td>11.55</td>
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<tr>
<td>( \text{Object B} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>7.707352</td>
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<td>14.227274</td>
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<td>14.227274</td>
</tr>
</tbody>
</table>

In our example (Fig. 1), according to formula (1)

\[
A = V \bigcup D, \quad B = H \bigcup K \quad \text{and} \quad n_A = n_B = 2.
\]

Assuming \( n' = n_A \cdot n_B \) we have non-overlapping constraint \( \Phi_{AB} \geq 0 \), where

\[
\Phi_{AB} = \min \{ \Phi_k : k = 1, 2, \ldots, n' = 4 \} = \min \{ \Phi_{VH}, \Phi_{VK}, \Phi_{DH}, \Phi_{DK} \},
\]

where \( \Phi_{VH}, \Phi_{VK}, \Phi_{DH}, \Phi_{DK} \) are phi-functions for basic objects \( V \) and \( H \), \( V \) and \( K \), \( D \) and \( H \), \( D \) and \( K \).

Using phi-functions the containment constraint \( (A \subset C_0, \quad B \subset C_0) \) has the form:

\[
\min \{ \Phi_{C_0^A}, \Phi_{C_0^B} \} \geq 0,
\]
Figure 1: a) Objects $A$ and $B$, b) Decomposition of $A$ and $A$ into basic objects, c) minimal enclosing rectangle, d) minimal enclosing circle

$$\Phi_{C^*A} = \min \{\Phi_{C^*_iA_i} : i = 1, 2, ..., n_A = 2\}, \quad \Phi_{C^*_B} = \min \{\Phi_{C^*_jB_j} : j = 1, 2, ..., n_B = 2\}.$$ Assuming $n'' = n_A + n_B$ we have

$$\min \{\Phi_{C^*_iT_k} : k = 1, 2, ..., n'' = 4, \ T_k \in \Theta\} = \min \{\Phi_{C^*_V}, \Phi_{C^*_D}, \Phi_{C^*_H}, \Phi_{C^*_K}\},$$

where $\Phi_{C^*_V}$, $\Phi_{C^*_D}$, $\Phi_{C^*_H}$, $\Phi_{C^*_K}$ are basic phi-functions for objects $C^*$ and $V$, $C^*$ and $D$, $C^*$ and $H$, $C^*$ and $K$.

## 5 General solution strategy

### 5.1 The branching tree

Since the considered problem has a non-convex disconnected feasible region $W$ we try to present $W$ as a union of subsets $W_s$, $s = 1, 2, ..., \eta$. Each $W_s$ is determined by a system of inequalities with infinitely differentiable functions extracted from our phi-functions. A phi-function of two objects $A$ and $B$ has in general the form

$$\Phi_{A,B}(u^A, \theta_A, u^B, \theta_B) = \min_{j} \max_{k} s_{jk}(u^A, \theta_A, u^B, \theta_B)$$
where the functions $f_{jk}$ can also contain min- and max-terms. Since $\min_j f_{jk}(x, y) \geq 0$ is equivalent to $f_{jk}(x, y) \geq 0$ for all $j$, and $\max_k f_{jk}(x, y) \geq 0$ means at least one of the inequalities, say $f_{j,k}(x, y) \geq 0$ has to be fulfilled, each of these terms can be considered as a system of (in general non-linear) inequalities (all max-terms are dissolved). This can be done by a branching scheme. It means that for each inequality $\Phi_{AB} \geq 0$ we may construct a tree, called phi-tree. To each terminal node of the phi-tree there corresponds a system of inequalities with infinitely differentiable functions.

Now we form a finite number of subproblems

$$F_{ks}(u^*_s) = \min \{F_k(u) : u \in W_s \} \quad (k \in \{1, 2, 3\}, \ s = 1, \ldots, \eta), \quad (3)$$

where $u = (u^A, \theta_A, u^B, \theta_B, p)$.

Clearly, the desired solution $u^*$ can be obtained by inspecting and exactly solving all of these subproblems, i.e.

$$u^* = \min \{u^*_s, s = 1, \ldots, \eta\}.$$ resulting subproblems are attacked by standard techniques (interior point method, feasible direction method) for local optimization.

The construction of all subproblems by means of a branching tree is illustrated by the following two examples.

**Example 2**

Let $D_i = \{(x, y) : g_i(x, y) \leq 0, h_i(x, y) \leq 0\}$ where $g_i(x, y) = (x - x_i)^2 + (y - y_i)^2 - r_i^2$ and $h_i(x, y) = a_i x + b_i y + c_i$ such that $\text{int}(D_i) \neq \emptyset$ and $h_i(x_i, y_i) > 0$, $i = 1, 2$, be circular segments with radii $r_i$, and let $C_0$ be a circular container of radius $r$. For simplicity, we assume $r < r_i$, $i = 1, 2$, so that the containment of $D_i$ within $C_0$ is guaranteed if the two intersection points $(x_{ij}, y_{ij})$, $j = 1, 2$ of $g_i$ and $h_i$ belong to $C_0$. The objective is to minimize $r$ so that $\text{int}(D_1) \cap D_2 = \emptyset$ and $D_i \subset C_0$, $i = 1, 2$. Then the mathematical model of the problem can be written in the form

$$F(u) = r \rightarrow \min \quad \text{s.t.} \quad u \in W \quad \text{(4)}$$

where $u = (r, x_1, y_1, \theta_1, x_2, y_2, \theta_2) \in IR^7$ and

$$W = \{u \in IR^7 : \Phi_{D_1D_2}(u) \geq 0, \ \Phi_{C_iD_i}(u) \geq 0, \ r \leq r_i, \ i = 1, 2\}.$$ The number of terminate nodes of the branching tree in this example equals $\eta = 19$ as we will see. In general, the number of leaves may be very large and therefore, a drawback of the basic approach. Further investigations are required to reduce it drastically. Moreover, the resulting subsets of $W$ are not disjunctive in general.

Due to the dissection of $W = \cup_{s=1}^\eta W_s$, problem (4) can be reduced to the following optimisation problem:

$$F(u^*) = \min \{F(u^*_s), F(u^*_s), \ldots, F(u^*_s)\} \quad \text{(5)}$$

where

$$F(u^*_s) = \min \{F(u) : u \in W_s\}, \ \ s = 1, 2, \ldots, \eta. \quad (6)$$
In order to construct subregions $W_s$ the dissection of $W$ is obtained by ensuring the validity of all restrictions. Note, in order to ensure $D_i(u) \subset C_0$ for $i = 1, 2$ the following constraints belong to each system of inequalities defining one of the sets $W_s$:

\[
 r^2 - (x_{ij} + x_i)^2 - (y_{ij} + y_i)^2 \geq 0, \quad j = 1, 2, \quad i = 1, 2.
\]  

(7)

Let us now consider restriction $\Phi_{D_1D_2}(u) \geq 0$. Since the two circular segments can be considered as intersection of two primary objects, a triangle and a circle, we have

\[
 D_1 = T_1 \cap C_1 \quad \text{and} \quad D_2 = T_2 \cap C_2
\]

where $T_1, T_2$ are appropriate triangles and $C_1, C_2$ corresponding circles. Hence,

\[
 \Phi_{D_1D_2} = \max\{\Phi_{D_1T_2}, \Phi_{D_1C_2}\},
\]

(8)

and moreover,

\[
 \Phi_{D_1T_2} = \max\{\Phi_{T_1T_2}, \Phi_{C_1T_2}\}, \quad \Phi_{D_1C_2} = \max\{\Phi_{C_1T_1}, \Phi_{C_1C_2}\}.
\]

(9)

Consequently, we have to consider phi-functions of primary objects $\Phi_{T_1T_2}, \Phi_{C_1T_2}, \Phi_{C_2T_1}$ and $\Phi_{C_1C_2}$.

Let $(x_i', y_i'), i = 1, 2, 3$, be the vertices of $T_1$, and $(x_i'', y_i''), i = 1, 2, 3$, those of $T_2$ (with $x_i' = x_{ij}, x_i'' = x_{2i}, i = 1, 2$). Furthermore, we suppose a representation of the two triangles as follows:

\[
 T_1 = \{(x, y) : \varphi_i = \alpha_i' x + \beta_i' y + \gamma_i' \leq 0, \quad i = 1, 2, 3\},
\]

\[
 T_2 = \{(x, y) : \psi_j = \alpha_j'' x + \beta_j'' y + \gamma_j'' \leq 0, \quad i = 1, 2, 3\}.
\]

Phi-function for two convex polygon $K_1$ and $K_2$ is defined as follows, e.g. Chernov, Stoyan and Romanova [23].

Let $(x_i', y_i'), i = 1, 2, \ldots, m_1$, be vertices of $K_1$, and $(x_i'', y_i''), j = 1, 2, \ldots, m_2$, be vertices of $K_2$, and $\varphi_i = \alpha_i' x + \beta_i' y + \gamma_i' = 0$ and $\psi_j = \alpha_j'' x + \beta_j'' y + \gamma_j'' = 0$ are side equations of $K_1$ and $K_2$, subject to $\varphi_i \leq 0$ and $\psi_j \leq 0$ for all points belonging to $K_1$ and $K_2$ respectively, then

\[
 \Phi_{K_1K_2} = \max\{\min_{1 \leq i \leq 3} \varphi_{ij}, \min_{1 \leq j \leq 3} \psi_{ji}\}.
\]

(10)

where $\varphi_{ij} = \alpha_i' x_j + \beta_i' y_j + \gamma_i'$, $\psi_{ji} = \alpha_j'' x_i + \beta_j'' y_i + \gamma_j''$ for all $i$ and $j$.

We define $\Phi_{T_1T_2} = \Phi_{K_1K_2}$ for $m_1 = m_2 = 3$.

For the two circles $C_i$ of radii $r_i$ and center points $(x_{C_i}, y_{C_i}), i = 1, 2$, we have

\[
 \Phi_{C_1C_2} = (x_{C_1} - x_{C_2})^2 + (y_{C_1} - y_{C_2})^2 - (r_1 + r_2)^2.
\]

(11)

Phi-functions $\Phi_{C_1T_2}$ and $\Phi_{C_2T_1}$ are defined as follows:

\[
 \Phi_{CT} = \max\{\chi_i, \min\{\omega_i, \psi_i\}, i = 1, 2, 3\},
\]

(12)

where

\[
 \chi_i = \alpha_i x + \beta_i y + \gamma_i - r \quad \text{with} \quad \alpha_i^2 + \beta_i^2 = 1,
\]

\[
 \omega_i = (x_C - x_i)^2 + (y_C - y_i)^2 - r^2,
\]

\[
 \psi_i = (\beta_{i-1} - \beta_i)(x_C - x_i) - (\alpha_{i-1} - \alpha_i)(y_C - y_i) + r(\alpha_{i-1}\beta_i - \alpha_i\beta_{i-1}).
\]
We use notations $\chi'_i, \omega'_i, \psi'_i$ for $\Phi_{C_1T_2}$ and $\chi''_i, \omega''_i, \psi''_i$ for $\Phi_{C_2T_1}$.

Since each phi-function is a composition of minima and maxima of smooth functions and taking into account relations (8)–(12), the branching tree (also called phi-tree) for $\Phi_{D_1D_2}(x_1, y_1, \theta_1, x_2, y_2, \theta_2) \geq 0$ has the form ($v_{11}$ is the root node, and $v_{ij}^{k-1}$ is the $j$-th node of $i$-th level of the tree which is an offspring of the $k$-th node of $i-1$-th level of the tree for constraint $\Phi_{D_1D_2} \geq 0$):

Level 1: $v_{11} \leftrightarrow \Phi_{D_1D_2} \geq 0$
Level 2: $v_{11}^{11} \leftrightarrow \Phi_{D_1T_2} \geq 0, v_{42}^{11} \leftrightarrow \Phi_{D_1C_2} \geq 0$
Level 3: $v_{31}^{41} \leftrightarrow \Phi_{T_1T_2} \geq 0, v_{32}^{41} \leftrightarrow \Phi_{C_1T_2} \geq 0, v_{33}^{41} \leftrightarrow \Phi_{T_1C_2} \geq 0, v_{34}^{41} \leftrightarrow \Phi_{C_1C_2} \geq 0$
Level 4: (19 nodes)
$v_{41}^{31} \leftrightarrow \{\varphi_{1j} \geq 0, j = 1, 2, 3\}, \; v_{32}^{41} \leftrightarrow \{\varphi_{2j} \geq 0, j = 1, 2, 3\}, \; v_{43}^{31} \leftrightarrow \{\varphi_{3j} \geq 0, j = 1, 2, 3\},$
$v_{44}^{31} \leftrightarrow \{\psi_{1i} \geq 0, i = 1, 2, 3\}, \; v_{45}^{31} \leftrightarrow \{\psi_{2i} \geq 0, i = 1, 2, 3\}, \; v_{46}^{31} \leftrightarrow \{\psi_{3i} \geq 0, i = 1, 2, 3\},$
$v_{42}^{32} \leftrightarrow \chi'_1 \geq 0, \; v_{43}^{32} \leftrightarrow \chi'_2 \geq 0, \; v_{49}^{32} \leftrightarrow \chi'_3 \geq 0, \; v_{44}^{32} \leftrightarrow \{\omega''_1 \geq 0, \omega''_1 \geq 0\},$
$v_{41}^{33} \leftrightarrow \{\omega''_2 \geq 0, \omega''_2 \geq 0\}, \; v_{42}^{33} \leftrightarrow \{\omega''_3 \geq 0, \omega''_3 \geq 0\}, \; v_{43}^{33} \leftrightarrow \{\omega''_4 \geq 0, \omega''_4 \geq 0\}, \; v_{44}^{33} \leftrightarrow \{\omega''_5 \geq 0, \omega''_5 \geq 0\}, \; v_{44}^{33} \leftrightarrow \{\omega''_6 \geq 0, \omega''_6 \geq 0\}, \; v_{44}^{33} \leftrightarrow \omega \geq 0.$

Thus the number of terminal nodes $\eta$ for $\Phi_{D_1D_2} \geq 0$ is 19. To each terminal node $v_k$ there corresponds a system of inequalities $A_k(u) \geq 0, k = 1, ..., 19.$

Example 3 is added into the paper just to demonstrate the approach for computing a global solution.

**Example 3**

Given two non-rotatable triangles, the task is to find an enclosing rectangle of minimal perimeter. Let $a, b$ be the variable length and width of enveloping rectangle, respectively, then the objective function is $F(u) = a + b$, and the vector of variables is $u = (a, b, x_1, y_1, x_2, y_2)$.

Polygon (triangle) $K_1$ has corner points $(2, -1), (0, 2)$ and $(-2, 0)$, and triangle $K_2$ has corner points $(0, 0), (3, 2)$ and $(0, 2)$ (cf. Fig. 2).

Fig. 2 shows the six cases to be considered in order to get a global minima (all subproblems are linear). The solution is $F^* = \min\{F(u_i), \; i = 1, \ldots, 6\} = F(u_4) = 7.666668$ with global minimum at point $u_4 = (4.000000, 3.666668, 2.000000, 1.000000, -0.000000, 1.666668)$.

### 5.2 The basic algorithm

The overall solution algorithm contains the following steps:

1. According to the branching scheme select one of the subproblems (3).
2. Find a starting point for subproblem (3).
3. Search for a local minimum for subproblem (3).
4. Search for the global minimum for subproblem (3) or an approximation of it:
5. If some termination criterion is met then stop, otherwise select a new subproblem of type (3) and continue with 2.
Figure 2: a) Polygons $K_1$ and $K_2$, b) Placement of $K_1$ and $K_2$ corresponding to point $u_i^*$ of a local minimum, $i = 1, \ldots, 6$
At this time, we consider the following realisations of subproblem (3):

1. Both objects $A$ and $B$ are (one-connected) polygons. Here we consider four problem classes: without or with rotations and linear or quadratic objective function. In the first problem class (P1) (without rotations, perimeter minimization) subproblem (3) becomes a linear problem. Class (P2) (without rotations, area minimization) has quadratic objective but linear constraints. In classes (P3) and (P4) (with rotations, perimeter or area minimization) non-linear constraints with sin- and cos-terms occur.

2. Objects $A$ and $B$ are composed objects. In similarity to above, four problem classes (P5) – (P8) are considered due to not allowing or allowing rotations and the two objective functions. In any case, non-linear (quadratic or with sin- and cos-terms) constraints (inequalities) are present.

In case, the branching tree is completely considered and all subproblems (3) are solved to optimality, we obtain a global minimum for problem classes (P1) and (P2).

6 Conclusions and Outlook

In the paper a basic approach is presented to handle placement problems with irregular shape. In particular, we consider objects whose frontiers formed by straight-line and/or circular-arc segments. We investigated the problem of enclosing two such objects by a rectangle or circle of minimal area or perimeter by means of phi-functions.

Since the number of subproblems to be solved increases rapidly when the objects become more complicated, pruning rules to reduce drastically that number are needed and will be part of future research.

7 Appendix

7.1 Appendix A: Basic phi-functions

Let $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}^2$ be phi-objects, bounded by circular arcs and line segments. We allow free rotations and translations of objects $A$ and $B$ in $\mathbb{R}^2$. Vector $u = (x_t, y_t, \theta)$ defines the arrangement of $A$ in $\mathbb{R}^2$. Hereby $\theta$ is the rotation parameter, and $(x_t, y_t)$ is the translation vector of $A$. Each point $(x_0, y_0) \in A$ in its eigen coordinate system is transformed into point $(x, y)$ according to

$$\begin{align*}
x &= x_0 \cdot \cos \theta + y_0 \cdot \sin \theta + x_t, \\
y &= -x_0 \cdot \sin \theta + y_0 \cdot \cos \theta + y_t,
\end{align*}$$

and each straight line $L_0 = \{(x, y) \in \mathbb{R}^2 \mid \alpha_0 x + \beta_0 y + \gamma_0 = 0, \sqrt{\alpha_0^2 + \beta_0^2} = 1 \}$ is transformed into straight line $L = \{(x, y) \in \mathbb{R}^2 \mid \alpha x + \beta y + \gamma = 0, \ where
\end{align*}$$

$$\begin{align*}
\alpha &= \alpha_0 \cdot \cos \theta + \beta_0 \cdot \sin \theta, \\
\beta &= -\alpha_0 \cdot \sin \theta + \beta_0 \cdot \cos \theta, \\
\gamma &= \gamma_0 - \alpha \cdot x_t - \beta \cdot y_t.
\end{align*}$$

Let us introduce a set of basic phi-objects:

$C$ is a circle, $C^* = \mathbb{R}^2 \setminus \text{int}(C)$ (cf. Fig. 3a) its complement, $K$ is a convex phi-polygon (cf.
Figure 3: Basic objects
Fig. 3b); \( K^* = \mathbb{R}^2 \setminus \text{int}(K) \) (cf. Fig. 3c), \( D = K \cap C \) is a circular segment (disk segment) (cf. Fig. 3d), \( H = C^* \cap K \) is a "hat" (cf. Fig. 3), \( V = G \cap W \) or \( V = H \cap W \) is a "horn" (cf. Fig. 3), and a set of auxiliary objects:

\( G = D \cap C^* \) (cf. Fig. 3), \( W = K' \cup D \) or \( W = K'' \cap C \) (cf. Fig. 3).

Remark 1. We suppose that two sides of \( K \) are tangents to the end points of the arc of objects \( D, H, W \) (cf. Fig. 3).

Remark 2. Each angle arc size of \( V \) is less then \( \pi \).

Now we give a complete class of basic radical-free phi-functions:

\[ \{ \Phi_{CC}, \Phi_{CK}, \Phi_{CD}, \Phi_{CH}, \Phi_{CV}, \Phi_{CC^*}, \Phi_{CK^*}, \Phi_{KD}, \Phi_{KH}, \Phi_{KV}, \Phi_{KK^*}, \Phi_{DD}, \Phi_{DH}, \Phi_{DV}, \Phi_{KC^*}, \Phi_{KK^*}, \Phi_{HH}, \Phi_{HV}, \Phi_{HC^*}, \Phi_{HK^*}, \Phi_{VV}, \Phi_{VC^*}, \Phi_{VK^*} \} \]

- **Objects \( C^* \) and \( C \)**, \( r_{C^*} \geq r_C \)

\[ \Phi_{C^*C} = -(x_{C^*} - x_C)^2 - (y_{C^*} - y_C)^2 + (r_{C^*} - r_C)^2. \] \hspace{1cm} (13)

- **Objects \( C^* \) and \( K \)**

\[ \Phi_{C^*K} = \min \{ \omega_i, i = 1, 2, \ldots, m \} \quad \text{with} \quad \omega_i = (x_{C^*} - x_i)^2 + (y_{C^*} - y_i)^2 - r^2. \] \hspace{1cm} (14)

- **Objects \( D \) and \( E \in \{ C, K, D \} \)**

\[ \Phi_{DE} = \max \{ \Phi_{CE}, \Phi_{KE} \}. \] \hspace{1cm} (15)

- **Objects \( K^* \) and \( D \)**

\[ \Phi_{K^*D} = \min \max_{i=1,\ldots,m} \{ \Phi_{K^*CD_i}, \Phi_{K^*KD_i} \}. \] \hspace{1cm} (16)

- **Objects \( C^* \) and \( D \)**

Let \( D = K \cap C_D \), where \( C_D \) is a circle of radius \( r_D \) with center \((x_D, y_D)\).

\[ \Phi_{C^*D} = \min \{ \psi_0, \max \{ \Phi_{C^*CD}, \chi_1, -\chi_2 \} \}, \] \hspace{1cm} (17)

where

\[ \psi_0 = \min_{i=1,2} \left( r_C^2 - (x_C - x_i)^2 - (y_C - y_i)^2 \right), \]

\[ \chi_i = (x_D - x_C)(y_i - y_D) - (y_D - y_C)(x_i - x_D), \quad i = 1, 2. \]

If \( r_D \geq r_C \) then we use \( \Phi_{C^*D} = \psi_0 \).

- **Objects \( W \) and \( E \in \{ C, K, D, P, W \} \)** \( W = K \cap C \) (cf. Fig. 3)

\[ \Phi_{WE} = \max \{ \Phi_{KE}, \Phi_{CE} \}. \] \hspace{1cm} (18)

- **Objects \( W \) and \( E \in \{ C^*, K^* \} \)** \( W = K \cup D \) (cf. Fig. 3)

\[ \Phi_{WE} = \min \{ \Phi_{KE}, \Phi_{DE} \}. \] \hspace{1cm} (19)
Figure 4: a) Objects $G$ and $C$, b) Objects $G$ and $K$ c) Objects $G$ and $D$ for $r_D < r_C$
• Objects \( G \) and \( C \)

\[
\Phi_{GC} = \max\{\Phi_{CGC}, \Phi_{PGC}, \min\{\omega_1, \psi_1, \omega_2, \psi_2\}\},
\]

where \( \omega_i = (x - x_i)^2 + (y - y_i)^2 - r^2 \), \( i = 1, 2 \), and \( \psi_i = 0 \) are equations of straight lines \( L_i \), passing through points \( t_{i1} \) and \( t_{i2} ; \psi_i(O_G) > 0, i = 1, 2, O_G \) is center point of \( C_G \) (Fig. 4a)).

• Objects \( G \) and \( K \)

\[
\Phi_{KG} = \min\{\Phi_{K1K}, \Phi_{K2K}, \psi\},
\]

where \( \psi = \min \max\{r_C^2 - (x_C - x_i)^2 - (y_C - y_i)^2, \alpha x_i + \beta y_i + \gamma\} \), \( r_C \) is radius of \( C = \mathbb{R}^2 \setminus \text{int}(C^*) \) (Fig. 4b)).

• Objects \( G \) and \( D \)

If \( r_D < r_C \) then phi-function of \( D \) and \( H \) is defined in the form (Fig. 4c))

\[
\Phi_{DG} = \max\{\Phi_{PC_D}, \Phi_{CG_D}, \Phi_{CG_D}, \Phi_{GD}, \varphi_1, \varphi_2\},
\]

here \( \varphi_1 = \min\{f_{P1CG}, f_{P2CG}, f_{P1PD}, f_{P2PD}, \Phi_{DK_2}\} \), \( \varphi_2 = \min\{f_{P2CG}, f_{P1PD}, f_{P2PD}, \Phi_{DK_1}\} \), i.e. \( \varphi_i = \min\{f_{P1CG}, f_{P2CG}, f_{P1PD}, f_{P2PD}, \Phi_{DK_1}\} \) where \( f_{P_iCG} \) is a function of belonging point \( p_i \) to \( C_G, + i = 1, 2 \), i.e. \( f_{P_iCG} = r_C^2 - (x_G - x_i)^2 - (y_G - y_i)^2 \), and where \( (x_1, y_1) \) and \( (x_2, y_2) \) are coordinates of endpoints \( p_1 \) and \( p_2 \) of \( D \); \( (x_G, y_G) \) are coordinates of center point of \( C_G \);

\( f_{P_iPD} \) is a function of belonging point \( t_i \) to \( P_D \) where \( P_D = \{\alpha_D x + \beta_D y + \gamma_D \leq 0\} \), \( L_D = \{(x, y) \in \mathbb{R}^2 : \alpha_D x + \beta_D y + \gamma_D = 0\} \), \( p_i \in L_D \), \( T_D \subset P_D, \ i = 1, 2 \).

If \( r_D \geq r_C \), then functions \( \varphi_1, \varphi_2 \) in (22) take the form

\[
\phi_1 = \min\{f_{P1CG}, f_{P2CG}, f_{P1PD}, f_{P2PD}, \Phi_{DK_2}, \chi_1\};
\]

\[
\phi_2 = \min\{f_{P2CG}, f_{P1PD}, f_{P2PD}, \Phi_{DK_1}, -\chi_2\};
\]

\[
\chi_i = (x_D - x_C)(y_i - y_D) - (y_D - y_C)(x_i - x_D), \ i = 1, 2.
\]

• Objects \( W \) and \( G \)

\( W = K \cup D \) (cf. Fig. 3)

\[
\Phi_{WG} = \min\{\Phi_{KG}, \Phi_{DG}\}.
\]

• Objects \( H \) and \( E \in \{C, K, D, P, W\} \)

\( H = K \cap G \)

\[
\Phi_{HE} = \max\{\Phi_{GE}, \Phi_{KE}\}.
\]

• Objects \( H \) and \( E \in \{C^*, K^*\} \)

\[
\Phi_{HE} = \Phi_{KE}.
\]
Objects $H$ and $H$

Let $H' = G' \cap T'$ be a hat and $H'' = G'' \cap T''$ another hat. Equivalently, $H' = (C')^* \cap T'$ and $H'' = (C'')^* \cap T''$. For the hat $H'$ we use notation $G', C', T'$, etc., as defined above, and for the hat $H''$ the respective notation $G'', C'', T''$, etc.

\[ \Phi^{H'H''} = \max \{ \Phi^{H''}, \Phi^{G'T''}, \omega, \tau \}, \]  
\[ \omega = \min \{ \alpha_2 x_i''' + \beta_2 y_i''' + \gamma_2', \alpha_2 x_i''' + \beta_2 y_i''' + \gamma_2', \]  
\[ (r_{C'})^2 - (x_c' - x_3')^2 - (y_c' - y_3')^2, \]  
\[ (r_{C''})^2 - (x_c'' - x_3'')^2 - (y_c'' - y_3'')^2 \}, \]

\[ \tau = \min \{ \alpha_1 x_i''' + \beta_1 y_i''' + \gamma_1', \alpha_1 x_i''' + \beta_1 y_i''' + \gamma_1', \]  
\[ (r_{C'})^2 - (x_c' - x_3')^2 - (y_c' - y_3')^2, \]  
\[ (r_{C''})^2 - (x_c'' - x_3'')^2 - (y_c'' - y_3'')^2 \}, \]

here $(x_i', y_i')$ are the coordinates of the vertices and $\alpha_i x + \beta_i y + \gamma_i = 0, i = 1, 2, \alpha_i x + \beta_i y + \gamma_i = 0, i = 1, 2$, are the equations of lines containing the two straight sides of $H'$, respectively; $(x_{C'}, y_{C'})$ and $r_{C'}$ are the coordinates of the center and the radius of the arc bounding $H'$. Similar notation apply to the hat $H''$.

Objects $V$ and $E \in \{ C, K, D, W \}$ $V = G \cap W$

\[ \Phi_{VE} = \max \{ \Phi_{GE}, \Phi_{WE} \}. \]  
\[ \Phi_{VE} = \min \{ \Phi_{DE}, f^{PE} \}, \]  
\[ \text{where } f^{PE} \text{ is a function of belonging point } p \text{ to } E^* = \mathbb{R}^2 \setminus \text{int}(E). \]

Objects $V$ and $H$

Let $V = H_V \cap W_V$, then

\[ \Phi_{HV} = \max \{ \Phi_{HH_V}, \Phi_{HW_V} \}. \]

Objects $V$ and $V$

Let $V_i = H_i \cap W_i, i = 1, 2$, then

\[ \Phi_{V_1V_2} = \max \{ \Phi_{H_1H_2}, \Phi_{W_2G_1}, \Phi_{W_1G_2}, \Phi_{W_1W_2} \}. \]

Formulas (11)-(30) complete the class of basic phi-functions.
7.2 Appendix B: Example 1

(a) Optimal covering of objects $A$ and $B$ by a circle

Point of the global minimum:
\[ u^* = (x^*, x_1^*, \theta_1^*, x_2^*, y_2^*) = (9.623201, -1.100537, -8.179520, 0, -0.310665, 5.370833, -2.808011) \]

\[ F(u^*) = r^* = 9.623201 \]

The system on which the best solution has been reached:

\[ -(7.7046 \sin \theta_2 + 7.1753 \cos \theta_2 + x_2)^2 - (7.1753 \sin \theta_2 + 7.7046 \cos \theta_2 + y_2)^2 + r_2 \geq 0 \]

\[ -(14.4198 \sin \theta_2 - 0.2176 \cos \theta_2 + x_2)^2 - (0.2176 \sin \theta_2 + 14.4198 \cos \theta_2 + y_2)^2 + r_2 \geq 0 \]

\[ -(6.7999 \sin \theta_2 - 4.2457 \cos \theta_2 + x_2)^2 - (4.2457 \sin \theta_2 + 6.7999 \cos \theta_2 + y_2)^2 + r_2 \geq 0 \]

\[ -(7.0743 \sin \theta_2 + 7.1665 \cos \theta_2 + x_2)^2 - (7.1665 \sin \theta_2 + 7.0743 \cos \theta_2 + y_2)^2 + r_2 \geq 0 \]

\[ -(7.0743 \sin \theta_2 - 4.2457 \cos \theta_2 + x_2)^2 - (4.2457 \sin \theta_2 + 7.0743 \cos \theta_2 + y_2)^2 + r_2 \geq 0 \]

\[ -(3.4453 \sin \theta_2 - 6.7145 \cos \theta_2 + x_2)^2 - (6.7145 \sin \theta_2 - 3.4453 \cos \theta_2 + y_2)^2 + r_2 \geq 0 \]

\[ -(x_1 - 3.5573)^2 - (y_1 - 0.1989)^2 + r_2 \geq 0 \]

\[ -(x_1 + 9.5178)^2 - (y_1 + 4.3682)^2 + r_2 \geq 0 \]

\[ 16.4329 \geq x_1 + 13.0751 \geq y_1 + 213.5205 \geq 0 \]

\[ -(x_1 + 7.7916)^2 - (y_1 + 13.4971)^2 + r_2 \geq 0 \]

\[ -(x_1 + 9.5239)^2 - (y_1 + 4.3741)^2 + r_2 \geq 0 \]

\[ -(x_1 + 7.7911)^2 - (y_1 + 13.5097)^2 + r_2 \geq 0 \]

\[ -(x_1 + 4.0898)^2 - (y_1 + 8.0755)^2 - 13.1500 \geq r + r_2 + 43.2306 \geq 0 \]

\[ -(7.0746 \sin \theta_2 + 7.1753 \cos \theta_2 + x_2)^2 - 2(7.0746 \sin \theta_2 + 7.1753 \cos \theta_2 + x_2) \geq (x_1 - 3.5573)^2 - (7.1753 \sin \theta_2 + 7.0746 \cos \theta_2 + y_2)^2 + 2(-7.1753 \sin \theta_2 + 7.0746 \cos \theta_2 + y_2) \geq (y_1 + 11.3511)^2 - (y_1 + 11.3511)^2 + 133.4025 \geq 0 \]

\[ -(14.4198 \sin \theta_2 - 0.2176 \cos \theta_2 + x_2)^2 - 2(14.4198 \sin \theta_2 - 0.2176 \cos \theta_2 + x_2) \geq (x_1 - 3.5573)^2 - (2(14.4198 \sin \theta_2 + 0.2176 \cos \theta_2 + x_2) \geq (y_1 + 11.3511)^2 - (y_1 + 11.3511)^2 + 133.4025 \geq 0 \]

\[ -(6.7999 \sin \theta_2 - 4.2457 \cos \theta_2 + x_2)^2 - 2(6.7999 \sin \theta_2 - 4.2457 \cos \theta_2 + x_2) \geq (x_1 - 3.5573)^2 - (6.7999 \sin \theta_2 + 4.2457 \cos \theta_2 + y_2) \geq (y_1 + 11.3511)^2 - (y_1 + 11.3511)^2 + 133.4025 \geq 0 \]

\[ -(7.0746 \sin \theta_2 + 7.1753 \cos \theta_2 + x_2)^2 - 2(y_1 + 9.5239)^2 + (-0.0004 \sin \theta_2 - \cos \theta_2)^2 \geq (y_1 + 4.3741)^2 - (y_1 + 9.5239)^2 + (0.0004 \sin \theta_2 + \cos \theta_2)^2 \geq r_2 + 7.7016 \geq 0 \]

\[ -(7.0746 \sin \theta_2 + 7.1753 \cos \theta_2 + x_2)^2 - 2(y_1 + 9.5239)^2 + (0.0004 \sin \theta_2 + \cos \theta_2)^2 \geq (y_1 + 9.5239)^2 - (y_1 + 9.5239)^2 + (0.0004 \sin \theta_2 - \cos \theta_2)^2 \geq r_2 + 7.7016 \geq 0 \]

\[ -(7.0746 \sin \theta_2 + 7.1753 \cos \theta_2 + x_2)^2 - 2(y_1 + 9.5239)^2 + (0.0004 \sin \theta_2 - \cos \theta_2)^2 \geq (y_1 + 9.5239)^2 - (y_1 + 9.5239)^2 + (0.0004 \sin \theta_2 + \cos \theta_2)^2 \geq r_2 + 7.7016 \geq 0 \]

\[ -(7.0746 \sin \theta_2 + 7.1753 \cos \theta_2 + x_2)^2 - 2(y_1 + 9.5239)^2 + (0.0004 \sin \theta_2 + \cos \theta_2)^2 \geq r_2 + 7.7016 \geq 0 \]

\[ -6385890.3280 \geq \sin \theta_2 - 76954796.6093 \geq \cos \theta_2 \geq 0 \]

\[ 88969116.2709 \geq \sin \theta_2 - 45656288.8902 \geq \cos \theta_2 \geq 0 \]

\[ -6385890.3280 \geq \sin \theta_2 - 76954796.6093 \geq \cos \theta_2 \geq 0 \]

\[ (y_1 - 0.1989)^2 - (y_1 - 0.1989)^2 + (0.0007 \sin \theta_2 + \cos \theta_2)^2 \geq 0 \]

\[ (x_1 - 3.5573)^2 - (x_1 - 3.5573)^2 + (0.0007 \sin \theta_2 + \cos \theta_2)^2 \geq 0 \]

\[ (0.2161 \sin \theta_2 - 0.9764 \cos \theta_2)^2 \geq (y_1 + 13.4971)^2 - (y_1 + 13.4971)^2 + (0.2161 \sin \theta_2 + 0.9764 \cos \theta_2)^2 \geq 0 \]

\[ -(7.6993 \sin \theta_2 - 4.2478 \cos \theta_2 + x_2)^2 + 2(7.6993 \sin \theta_2 - 4.2478 \cos \theta_2 + x_2)^2 \geq (x_1 - 3.5573)^2 - (7.6993 \sin \theta_2 + 4.2478 \cos \theta_2 + y_2)^2 + 2(7.6993 \sin \theta_2 + 4.2478 \cos \theta_2 + y_2)^2 \geq (y_1 + 11.3511)^2 - (y_1 + 11.3511)^2 + 133.4025 \geq 0 \]

\[ -(7.6993 \sin \theta_2 - 4.2478 \cos \theta_2 + x_2)^2 + 2(7.6993 \sin \theta_2 - 4.2478 \cos \theta_2 + x_2)^2 \geq (x_1 - 3.5573)^2 - (7.6993 \sin \theta_2 + 4.2478 \cos \theta_2 + y_2)^2 + 2(7.6993 \sin \theta_2 + 4.2478 \cos \theta_2 + y_2)^2 \geq (y_1 + 11.3511)^2 - (y_1 + 11.3511)^2 + 133.4025 \geq 0 \]
b) Optimal clustering objects $A$ and $B$ into a rectangle $R$

Point of the global minimum: $u^* = (a^*, b^*, x_{1}^{*}, y_{1}^{*}, \theta_1, x_{2}^{*}, y_{2}^{*}, \theta_2) =$

$((19.566388, 10.054259, 15.403537, 2.613369, 4.020905, 7.311720, 2.451621, 7.313793)\)

$F(u^*) = 196.725538$

The system on which the best solution has been reached:

$7.7046 \sin \theta_2 + 7.1753 \cos \theta_2 + x_2 \geq 0$

$14.4198 \sin \theta_2 - 0.2176 \cos \theta_2 + x_2 \geq 0$

$7.6999 \sin \theta_2 - 4.2457 \cos \theta_2 + x_2 \geq 0$

$-7.1753 \sin \theta_2 + 7.7046 \cos \theta_2 + y_2 \geq 0$

$0.2176 \sin \theta_2 + 14.4198 \cos \theta_2 + y_2 \geq 0$

$4.2457 \sin \theta_2 + 7.6999 \cos \theta_2 + y_2 \geq 0$

$7.7074 \sin \theta_2 + 7.1665 \cos \theta_2 + x_2 \geq 0$

$7.6993 \sin \theta_2 - 4.2478 \cos \theta_2 + x_2 \geq 0$

$-3.4453 \sin \theta_2 - 6.7145 \cos \theta_2 + x_2 \geq 0$

$-7.1665 \sin \theta_2 + 7.7074 \cos \theta_2 + y_2 \geq 0$

$4.2478 \sin \theta_2 + 7.6993 \cos \theta_2 + y_2 \geq 0$

$6.7145 \sin \theta_2 - 3.4453 \cos \theta_2 + y_2 \geq 0$

$-7.7046 \sin \theta_2 - 7.1753 \cos \theta_2 - x_2 + a \geq 0$

$-14.4198 \sin \theta_2 + 0.2176 \cos \theta_2 - x_2 + a \geq 0$

$-7.6999 \sin \theta_2 + 4.2457 \cos \theta_2 - x_2 + a \geq 0$

$7.1753 \sin \theta_2 - 7.7046 \cos \theta_2 - y_2 + b \geq 0$

$-0.2176 \sin \theta_2 - 14.4198 \cos \theta_2 - y_2 + b \geq 0$

$-4.2457 \sin \theta_2 - 7.6999 \cos \theta_2 - y_2 + b \geq 0$

$-7.7074 \sin \theta_2 - 7.1665 \cos \theta_2 - x_2 + a \geq 0$

$-7.6993 \sin \theta_2 + 4.2478 \cos \theta_2 - x_2 + a \geq 0$

$3.4453 \sin \theta_2 + 6.7145 \cos \theta_2 - x_2 + a \geq 0$

$7.1665 \sin \theta_2 - 7.7074 \cos \theta_2 - y_2 + b \geq 0$

$-4.2478 \sin \theta_2 - 7.6993 \cos \theta_2 - y_2 + b \geq 0$

$-6.7145 \sin \theta_2 + 3.4453 \cos \theta_2 - y_2 + b \geq 0$

$-0.1989 \sin \theta_1 - 3.5573 \cos \theta_1 + x_1 \geq 0$

$-0.1989 \sin \theta_1 + 3.7779 \cos \theta_1 + x_1 \geq 0$

$4.3682 \sin \theta_1 + 9.5178 \cos \theta_1 + x_1 \geq 0$

$3.5573 \sin \theta_1 - 0.1989 \cos \theta_1 + y_1 \geq 0$

$-3.7779 \sin \theta_1 - 0.1989 \cos \theta_1 + y_1 \geq 0$

$-9.5178 \sin \theta_1 + 4.3682 \cos \theta_1 + y_1 \geq 0$

$13.4971 \sin \theta_1 + 7.7916 \cos \theta_1 + x_1 \geq 0$

$-7.7916 \sin \theta_1 + 13.4971 \cos \theta_1 + y_1 \geq 0$

$8.0755 \sin \theta_1 + 4.0898 \cos \theta_1 + x_1 - 6.5750 \geq 0$

$-9.5239 \sin \theta_1 + 4.3741 \cos \theta_1 + y_1 \geq 0$

$-13.2253 \sin \theta_1 + 9.8083 \cos \theta_1 + y_1 \geq 0$

$-7.7911 \sin \theta_1 + 13.5097 \cos \theta_1 + y_1 \geq 0$

$0.1989 \sin \theta_1 + 3.5573 \cos \theta_1 - x_1 + a \geq 0$

$0.1989 \sin \theta_1 - 3.7779 \cos \theta_1 - x_1 + a \geq 0$

$-4.3682 \sin \theta_1 - 9.5178 \cos \theta_1 - x_1 + a \geq 0$

$-3.5573 \sin \theta_1 - 20.8011 \cos \theta_1 - y_1 + a - 21 \geq 0$

$-13.4971 \sin \theta_1 - 7.7916 \cos \theta_1 - x_1 + a \geq 0$

$7.7916 \sin \theta_1 - 13.4971 \cos \theta_1 - y_1 + 21 \geq 0$

$-4.3741 \sin \theta_1 - 9.5239 \cos \theta_1 - x_1 + a \geq 0$

$-9.8083 \sin \theta_1 - 13.2253 \cos \theta_1 - x_1 + a \geq 0$

$-13.5097 \sin \theta_1 - 7.7911 \cos \theta_1 - x_1 + a \geq 0$
\[9.5239 \sin \theta_1 - 4.3741 \cos \theta_1 - y_1 + b \geq 0\]

\[13.2253 \sin \theta_1 - 9.8083 \cos \theta_1 - y_1 + b \geq 0\]

\[7.7911 \sin \theta_1 - 13.5097 \cos \theta_1 - y_1 + b \geq 0\]

\[(-\sin \theta_1) * (7.7046 \sin \theta_2 + 7.1753 \cos \theta_2 + x_2) + (-\cos \theta_1) * (-7.1753 \sin \theta_2 + 7.7046 \cos \theta_2 + y_2) - (-\sin \theta_1) * x_1 + (-\cos \theta_1) * y_1 - 0.1989 \geq 0\]

\[(-\sin \theta_1) * (14.4198 \sin \theta_2 - 0.2176 \cos \theta_2 + x_2) + (-\cos \theta_1) * (0.2176 \sin \theta_2 + 14.4198 \cos \theta_2 + y_2) - (-\sin \theta_1) * x_1 + (-\cos \theta_1) * y_1 - 0.1989 \geq 0\]

\[(-\sin \theta_1) * (7.6999 \sin \theta_2 - 4.2457 \cos \theta_2 + x_2) + (-\cos \theta_1) * (4.2457 \sin \theta_2 + 7.6999 \cos \theta_2 + y_2) - (-\sin \theta_1) * x_1 + (-\cos \theta_1) * y_1 - 0.1989 \geq 0\]

\[(-\sin \theta_2 + 0.0004 \cos \theta_2) * (4.3741 \sin \theta_1 + 9.5239 \cos \theta_1 + x_1) + (0.0004 \sin \theta_2 - \cos \theta_2) * (-9.5239 \sin \theta_1 + 4.3741 \cos \theta_1 + y_1) - (-\sin \theta_2 + 0.0004 \cos \theta_2) * x_2 + (0.0004 \sin \theta_2 - \cos \theta_2) * y_2 + 7.7016 \geq 0\]

\[(-\sin \theta_2 + 0.0004 \cos \theta_2) * (9.8083 \sin \theta_1 + 13.2253 \cos \theta_1 + x_1) + (0.0004 \sin \theta_2 - \cos \theta_2) * (-13.2253 \sin \theta_1 + 9.8083 \cos \theta_1 + y_1) - (-\sin \theta_2 + 0.0004 \cos \theta_2) * x_2 + (0.0004 \sin \theta_2 - \cos \theta_2) * y_2 + 7.7016 \geq 0\]

\[100000000 * (-0.9441 * \sin \theta_1 + 0.3298 * \cos \theta_1) * (0.6263 * \sin \theta_2 + 0.7796 * \cos \theta_2) - 100000000 \geq 0\]

\[100000000 * (0.1858 * \sin \theta_1 + 0.9826 * \cos \theta_1) * (0.6263 * \sin \theta_2 + 0.7796 * \cos \theta_2) + 100000000 \geq 0\]

\[(-0.9441 * \sin \theta_1 + 0.3298 * \cos \theta_1) * (7.7074 \sin \theta_2 + 7.1665 \cos \theta_2 + x_2) + (-0.3298 * \sin \theta_1 - 0.9441 * \cos \theta_1) * (-7.1665 \sin \theta_2 + 7.7074 \cos \theta_2 + y_2) - (0.9441 * \sin \theta_1 + 0.3298 * \cos \theta_1) * x_1 + (-0.3298 * \sin \theta_1 - 0.9441 * \cos \theta_1) * y_1 + 0.9853 \geq 0\]

\[0.9853 \geq 0\]

\[0.9853 \geq 0\]

\[0.9853 \geq 0\]
\[-7.1766 \sin \theta_2 - 6.5199 \cos \theta_2 + y_2 \right)^2 + 202.4153 \geq 0 \]
\[(8.0755 \sin \theta_1 + 4.0898 \cos \theta_1 + x_1) \cdot (-5.4342 \sin \theta_1 - 3.7014 \cos \theta_1) - (-4.0898 \sin \theta_1 + 8.0755 \cos \theta_1 + y_1) \cdot (-3.7014 \sin \theta_1 + 5.4342 \cos \theta_1) - (-5.4342 \sin \theta_1 - 3.7014 \cos \theta_1) \cdot (-6.5199 \sin \theta_2 + 7.1766 \cos \theta_2 + x_2) + (-3.7014 \sin \theta_1 + 5.4342 \cos \theta_1) \cdot (-7.1766 \sin \theta_2 - 6.5199 \cos \theta_2 + y_2) \geq 0 \]
References


