Improved Upper Bounds for the Gap of the Skiving Stock Problem

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Improved Upper Bounds for the Gap of the Skiving Stock Problem

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Abstract

We consider the one-dimensional skiving stock problem (SSP) which is strongly related to the dual bin packing problem. In the classical formulation, different (small) item lengths and corresponding availabilities are given. We aim at maximizing the number of objects with a certain minimum length that can be constructed by connecting the items on hand. Such computations are of high interest in many real world application, e.g. in industrial recycling processes, wireless communications and politico-economic questions. Since the SSP is known to be NP-hard a common solution approach consists in solving an LP-based relaxation and the application of (appropriate) heuristics. Practical experience and computational simulations have shown that there is only a small difference (called gap) between the optimal objective values of the relaxation and the SSP itself. In this paper, we aim at evaluating the quality of the continuous relaxation by providing some new results and improved upper bounds for the gap of an arbitrary instance $E = (m, l, L, b)$. To this end, we introduce and mainly refer to the theory of residual instances.

Key words: Skiving Stock Problem, Gap, Continuous Relaxation, Proper Relaxation

1 Introduction and Preliminaries

In this paper, we consider the one-dimensional skiving stock problem (SSP) which is strongly related to the dual bin packing problem (DBPP). In the classical formulation, $m \in \mathbb{N} := \{1, 2, \ldots\}$ different item lengths $l_1, \ldots, l_m$ with availabilities $b_1, \ldots, b_m$ are given, the so-called items supply. We aim at maximizing the number of products with minimum length $L$ that can be constructed by connecting the items on hand.

Such computations are of high interest in many real world applications, e.g. industrial production processes (see [Zak03] for an overview) or politico-economic problems (cf. [AJKL84] and [LLM95]). Furthermore, also neighboring tasks, such as dual vector packing problems [CFGR91] or the maximum cardinality bin packing problem [BD85], [PD06], are often associated or even identified with the dual bin packing problem. These formulations are of
practical use as well since they are applied in multiprocessor scheduling problems [ARGA04] or surgical case planners [VPS13].

Throughout this paper, we will use the abbreviation $E := (m, l, L, b)$ for an instance of the SSP with $l = (l_1, \ldots, l_m)^\top$ and $b = (b_1, \ldots, b_m)^\top$. Without loss of generality, we assume all input-data to be positive integers with $L > l_1 > \ldots > l_m > 0$.

The classical solution approach is due to [Zak03] and based on the formulation of Gilmore and Gomory in the context of one-dimensional cutting [GG61]. Any feasible arrangement of items leading to a final product of minimum length is called a (packing) pattern of $E$. We always represent a pattern by a nonnegative vector $a = (a_1, \ldots, a_m)^\top \in \mathbb{Z}_+^m$ where $a_i \in \mathbb{Z}_+$ denotes the number of items of type $i \in I := \{1, \ldots, m\}$ being contained in the considered pattern. For a given instance $E$, the set of all patterns is defined by $P(E) := \{a \in \mathbb{Z}_+^m \mid l^\top a \geq L\}$. In [MS14], the authors slightly improve Zak’s formulation by only considering so-called minimal patterns obtaining a finite model, hereinafter referred to as the standard model, of the skiving stock problem. A pattern $a \in P(E)$ is called minimal if there exists no pattern $\tilde{a} \in P(E)$ such that $\tilde{a} \neq a$ and $\tilde{a} \leq a$ hold (componentwise). The set of all minimal patterns is denoted by $P^*(E)$. Furthermore, let $x_j \in \mathbb{Z}_+$ denote the number how often the minimal pattern $a^j = (a_{ij}, \ldots, a_{mj})^\top \in \mathbb{Z}_+^m$ ($j \in J^*$) of $E$ is used, where $J^* = \{1, \ldots, n\}$ represents an index set of all minimal patterns. Then the skiving stock problem can be formulated as

$$z^*(E) = \max \left\{ \sum_{j \in J^*} x_j \mid \sum_{j \in J^*} a_{ij} x_j \leq b_i, \ i \in I, \ x_j \in \mathbb{Z}_+, \ j \in J^* \right\}.$$

A common (approximate) solution approach consists in considering the continuous relaxation

$$z^*_c(E) = \max \left\{ \sum_{j \in J^*} x_j \mid \sum_{j \in J^*} a_{ij} x_j \leq b_i, \ i \in I, \ x_j \geq 0, \ j \in J^* \right\}$$

and the application of appropriate heuristics.

Practical experience and computational simulations, cf. [Zak03], have shown that there is only a small gap $\Delta(E) := z^*_c(E) - z^*(E)$ for any instance $E$. Based on the contributions of Baum and Trotter [BTS11] for general linear maximization problems, these observations have initiated the following definitions. A set $\mathcal{T}$ of instances has the integer round-down property (IRDP), if $\Delta(E) < 1$ holds for all $E \in \mathcal{T}$, and it has the modified integer round-down property (MIRDP), if $\Delta(E) < 2$ holds for all $E \in \mathcal{T}$. Whenever $\mathcal{T} = \{E\}$ is a singleton, we briefly say that $E$, instead of $\{E\}$, has the (modified) integer round-down property. An instance $E$ with $\Delta(E) \geq 1$ is called non-IRDP instance. It is conjectured, cf. [Zak03], that the one-dimensional skiving stock problem possesses the MIRDP. Currently, to our best knowledge, the best upper bound has been derived in [MS15] and is given by $\Delta(E) < m - 1$ for an instance $E = (m, l, L, b)$.

In the next sections, we introduce the theory of residual instances and show how it can be applied to obtain stronger upper bounds. Afterwards, we present two alternative approaches to evaluate the gap of a given instance. In a final step, we give some conclusions and an outlook of future research.
2 An Introduction to Residual Instances

In this paper, we study the difference between the optimal objective values of the skiving stock problem

\[
z^*(E) = \max \left\{ \sum_{j \in J^*} x_j \left| \sum_{j \in J^*} a_{ij} x_j \leq b_i, i \in I, x_j \in \mathbb{Z}_+, j \in J^* \right. \right\}
\]

(1)

and its corresponding continuous relaxation

\[
z^*_c(E) = \max \left\{ \sum_{j \in J^*} x_j \left| \sum_{j \in J^*} a_{ij} x_j \leq b_i, i \in I, x_j \geq 0, j \in J^* \right. \right\}.
\]

(2)

**Definition 1.** Let \( E = (m, l, L, b) \) be an instance of the skiving stock problem. Then, the difference

\[
\Delta(E) := z^*_c(E) - z^*(E)
\]

(3)

is called gap (of \( E \)).

In what follows, we show how the calculation of the gap of an instance \( E = (m, l, L, b) \) can be reformulated in some sense. To this end, we replace \( E \) by an instance \( \overline{E} = (m, l, \overline{L}, \overline{b}) \) with principally the same input-data, but a (appropriately) reduced vector \( \overline{b} \) of availabilities, and aim at computing the gap \( \Delta(\overline{E}) \). Hence, due to the smaller number of objects, the optimization problems becomes more manageable in a certain extent. On the other hand, this new gap \( \Delta(\overline{E}) \) only provides an upper bound for \( \Delta(E) \). As an initial point, we describe the construction of \( \overline{E} \) more precisely.

**Definition 2.** Let \( E = (m, l, L, b) \) be an instance of the skiving stock problem, and let \( x^c \) denote a solution of the corresponding continuous relaxation (2). Then the instance

\[
\overline{E} := \overline{E}(x^c) := (m, l, L, b - A \lceil x^c \rceil)
\]

is called residual instance of \( E \).

In the following, let \( A(E) \) (or briefly \( A \)) denote the matrix whose columns are equal to the elements of \( P^*(E) \), i.e., to the minimal patterns of the given instance \( E \). Then we state the following general result.

**Lemma 1.** Let \( E = (m, l, L, b) \) be an instance of the skiving stock problem, and let \( x^c \) denote a solution of the continuous relaxation (2) of \( E \). Then \( Ax^c = b \) holds.

**Proof.** For the sake of contradiction, we assume that there exists \( i \in I \) with

\[
b_i - [Ax^c]_i > 0.
\]

Let \( \tau \in J^* \) denote the index that corresponds to the elementary pattern \( a^\tau = k_i \cdot e^i \) with \( k_i := \lceil L/l_i \rceil \) and \( e^i \) as the \( i \)-th unit vector. Then, we define

\[
\overline{x}_j := \begin{cases} x^c_j & \text{for } j \in J^* \setminus \{\tau\}, \\ x^c_j + \frac{b_i - [Ax^c]_i}{k_i} & \text{for } j = \tau. \end{cases}
\]
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Since
\[ \sum_{j \in J^*} a_{ij} \tilde{x}_j = \sum_{j \in J^*} a_{ij} x_j^c + k_i \cdot \frac{b_i - [Ax^c]_i}{k_i} = [Ax^c]_i + b_i - [Ax^c]_i = b_i \]
is true and the definition of \( \tilde{x} \) does not affect the other constraints, we have \( A\tilde{x} \leq b \), i.e., \( \tilde{x} \) is a feasible point for (2). But now
\[ \sum_{j \in J^*} \tilde{x}_j = \sum_{j \in J^*} x_j^c + \frac{b_i - [Ax^c]_i}{k_i} > \sum_{j \in J^*} x_j^c, \]
so \( x \) is not a solution of (2) which gives the contradiction.

Note that the system matrix \( A \) is the same for the original instance \( E \) and each of its residual instances \( E(x^c) \). We start with a first result concerning the optimal objective values \( z^*_c(E) \) and \( z^*_c(E(x^c)) \) of the continuous relaxations of \( E \) and \( E(x^c) \), respectively.

**Lemma 2.** Let \( E = E(x^c) \) be a residual instance of \( E \), then \( z^*_c(E(x^c)) - z^*_c(E) \in \mathbb{Z}_+ \).

**Proof.** We define \( \overline{x} := \lfloor x^c \rfloor \) and \( \overline{x} := x^c - \overline{x} \). Then, obviously, \( 0 \leq \overline{x} < e \) holds where \( e \) denotes an appropriate-sized vector whose entries are all equal to 1. Additionally, \( \overline{x} \) is a feasible solution of the continuous relaxation of \( E \) since
\[ A\overline{x} = A(x^c - \overline{x}) = Ax^c - A\lfloor x^c \rfloor = b - A\lfloor x^c \rfloor = \overline{b} \]
is true due to \( Ax^c = b \) from the previous lemma.

Actually, \( \overline{x} \) is a solution of this relaxation: For the sake of contradiction, we assume that there exists \( \overline{x} \geq 0 \) with \( A\overline{x} = b - A\lfloor x^c \rfloor \) and \( e^\top \overline{x} > e^\top \overline{x} \). Then we have
\[ A(\overline{x} + \lfloor x^c \rfloor) = (b - A\lfloor x^c \rfloor) + A\lfloor x^c \rfloor = b, \]
i.e., \( \overline{x} + \lfloor x^c \rfloor \) is feasible for the continuous relaxation of \( E \). Furthermore, we obtain
\[ e^\top (\overline{x} + \lfloor x^c \rfloor) = e^\top \overline{x} + e^\top \lfloor x^c \rfloor > e^\top \overline{x} + e^\top \lfloor x^c \rfloor = e^\top (\overline{x} + \lfloor x^c \rfloor) \]
\[ = e^\top x^c \geq e^\top (\overline{x} + \lfloor x^c \rfloor), \]
since \( x^c \) is a solution which gives the contradiction.

Consequently, we have
\[ z^*_c(E(x^c)) - z^*_c(E) = e^\top \overline{x} = e^\top (x^c - \overline{x}) = z^*_c(E(x^c)) - e^\top \overline{x} \in \mathbb{Z}_+ \]
and the proof is complete.

In a second step, we want to investigate the relation between the gap of an instance and any of its residual instances.

**Lemma 3.** Let \( E = E(x^c) \) be a residual instance of \( E \), then \( \Delta(E) \leq \Delta(E(x^c)) \).
Proof. Let $\bar{x}^*$ be a solution of the instance $\bar{E}$. Then, $\bar{x}^* + \bar{x}$ is feasible for $E$ due to
\[ A(\bar{x}^* + \bar{x}) = A(\bar{x}^* + \lfloor x \rfloor) \leq \bar{b} + A[x] = b - A[x] + A[x] = b \]
and $\bar{x}^* + \bar{x} \in Z_m^+$. Furthermore, we have
\[ z^*(E) \geq e^\top (\bar{x}^* + \bar{x}) = z^*(\bar{E}) + e^\top \bar{x}. \]
With this inequality and the previous lemma, we obtain
\[
\Delta(E) = z^*_c(\bar{E}) - z^*(E) \leq z^*_c(\bar{E}) - z^*(\bar{E}) - e^\top \bar{x} = z^*_c(\bar{E}) + e^\top \bar{x} - z^*(\bar{E}) - e^\top \bar{x} = z^*_c(\bar{E}) - z^*(\bar{E}) = \Delta(\bar{E})
\]
and the proof is complete.

Hence, the gap of an instance $E$ can be bounded above by the gap of a corresponding residual instance $\bar{E}$. In particular, any upper bound for the gap $\Delta(\bar{E})$ of a residual instance (of $E$) is also applicable to the original gap $\Delta(E)$. This leads to the following observations.

**Corollary 4.** Let $\bar{E} = \bar{E}(x^c)$ be a residual instance of $E$ then:

1. If $\bar{E}$ has the IRDP, then $E$ has the IRDP.
2. If $\bar{E}$ has the MIRDP, then $E$ has the MIRDP.

**Proof.** Both assertions are direct implications of Lemma 3.

Thus, the consideration of residual instances is sufficient, in general.

### 3 Improved Upper Bounds

In this section, we aim at applying the theory of residual instances in order to obtain upper bounds of better quality for the gap of the skiving stock problem. To this end, the following lemma will prove beneficial.

**Lemma 5.** Let $\bar{E}$ be a residual instance (of $E$) and $p, q \in \mathbb{R}$ with $p \geq 1$. Then, the implication
\[
z^*_c(\bar{E}) < p \cdot z^*(\bar{E}) + q \quad \Rightarrow \quad \Delta(\bar{E}) < \frac{p - 1}{p} \cdot m + \frac{q}{p} = m \left( 1 - \frac{1}{p} \right) + \frac{q}{p}
\]
holds.

**Proof.** Let $x$ be a solution of the continuous relaxation of $\bar{E}$. The corresponding basis matrix consists of at most $m$ different patterns, i.e., at most $m$ components of $x$ are positive. Since $\bar{E}$ is a residual instance, without loss of generality, we can assume $0 \leq x < e$ (componentwise). Hence, we obtain
\[
\Delta(\bar{E}) = z^*_c(\bar{E}) - z^*(\bar{E}) < m - z^*(\bar{E})
\]
and the assumption of this lemma leads to
\[
\Delta(E) < (p - 1)z^*(E) + q. \tag{6}
\]
If we multiply (5) with \(p - 1\) and add (6), we obtain
\[
\Delta(E) < \frac{p - 1}{p} \cdot m + \frac{q}{p}
\]
which completes the proof.

As a first application, the concept of residual instances provides a simple proof and a slight generalization of [MS15, Theorem 1].

**Theorem 6.** Let \(E = (m, l, L, b)\) be an instance of the skiving stock problem, then \(\Delta(E) < \max\{1, m - 1\}\).

**Proof.** Let \(\overline{E} = \overline{E}(x^c)\) denote a residual instance of \(E\). Due to Lemma 3 it suffices to prove the assertion for the gap \(\Delta(\overline{E})\). If \(z^*(\overline{E}) \geq 1\) holds, the claim follows immediately from (5) since
\[
\Delta(\overline{E}) = z_c^*(\overline{E}) - z^*(\overline{E}) < m - z^*(\overline{E}) \leq m - 1.
\]
Otherwise, if \(z^*(\overline{E}) < 1\) is true, we obtain \(z^*(\overline{E}) = 0\) due to the integrality of the optimal objective value. Obviously, this implies
\[
l^\top \overline{b} < L
\]
and therefore \(z_c^*(\overline{E}) < 1\). Altogether, this results to
\[
\Delta(\overline{E}) = z_c^*(\overline{E}) - z^*(\overline{E}) < 1 - 0 = 1
\]
which completes the proof.

A similar, but more powerful, result can be obtained as follows.

**Theorem 7.** Let \(E = (m, l, L, b)\) be an instance of the skiving stock problem, then \(\Delta(E) < \max\{2, m - 2\}\).

**Proof.** Let \(\overline{E} = \overline{E}(x^c)\) denote a residual instance of \(E\). Due to Lemma 3 it suffices to prove the assertion for the gap \(\Delta(\overline{E})\). If \(z^*(\overline{E}) \geq 2\) holds, the claim follows immediately from (5) due to
\[
\Delta(\overline{E}) = z_c^*(\overline{E}) - z^*(\overline{E}) < m - z^*(\overline{E}) \leq m - 2.
\]
Otherwise, the two cases \(z^*(\overline{E}) \in \{0, 1\}\) are left. Obviously, \(z^*(\overline{E}) = 0\) holds if and only if \(l^\top \overline{b} < L\). This case is contained in the previous proof and leads to
\[
\Delta(\overline{E}) < 1 < \max\{2, m - 2\}.
\]
If \(z^*(\overline{E}) = 1\) is true, we have \(l^\top \overline{b} < 3L\) as a necessary condition since, otherwise, it would be possible to build at least two disjoint patterns with the given items. Consequently, we obtain \(z_c^*(\overline{E}) < 3\) and
\[
\Delta(\overline{E}) = z_c^*(\overline{E}) - z^*(\overline{E}) < 3 - 1 = 2
\]
which completes the proof.
In particular, this observation has an important implication.

**Corollary 8.** Let \( E = (m, l, L, b) \) be an instance with \( m = 4 \) item types. Then \( E \) has the MIRDP.

**Proof.** This follows directly from Theorem 7 with \( m = 4 \).

In the two previous lemmas we did not apply Lemma 5 directly, but only a part of its proof. For almost all values of \( m \), the following theorem improves the upper bound of the gap significantly by using the sufficient condition in (4).

**Theorem 9.** Let \( E = (m, l, L, b) \) be an instance of the skiving stock problem, then \( \Delta(E) < (m + 1)/2 \).

**Proof.** Let \( \overline{E} = \overline{E}(x^c) \) denote a residual instance of \( E \). Due to Lemma 3 it suffices to prove the assertion for the gap \( \Delta(\overline{E}) \). Obviously, each minimal pattern \( a \in P^*(\overline{E}) = P^*(\overline{E}) \) satisfies \( l^\top a < 2L \). If we apply the first phase of the FFD heuristic for the dual bin packing problem to \( \overline{E} \) with bin size \( l_B = 2L - 1 \) (cf. [MS15]), we obtain an allocation where all but possibly the last bin correspond to a pattern of \( \overline{E} \). Let \( q \in \mathbb{N} \) denote the number of nonempty bins after the first phase and let \( a_{ip} \in \mathbb{Z}_+ \) \((i \in I, p \in \{1, \ldots, q\})\) denote the number of items of type \( i \) in the \( p \)-th bin. Then, we have to consider two cases:

i) If the last bin contains objects whose lengths sum up to at least \( L \), i.e., if \( l^\top a_q \geq L \) holds, we have found a feasible solution with objective value \( q \). Then we have

\[
z^*_c(\overline{E}) \leq \frac{l^\top \overline{b}}{L} = \frac{l^\top a_j}{L} < 2q \leq 2z^*(\overline{E}) \implies z^*_c(\overline{E}) < 2 \cdot z^*(\overline{E}).
\]

ii) If the last bin does not contain objects whose lengths sum up to at least \( L \), i.e., if \( l^\top a_q < L \) holds, we have found a feasible solution with objective value \( q - 1 \). Then, we have

\[
z^*_c(\overline{E}) \leq \frac{l^\top \overline{b}}{L} = \frac{l^\top a_j}{L} < 2(q - 1) + 1 \leq 2z^*(\overline{E}) + 1 \implies z^*_c(\overline{E}) < 2 \cdot z^*(\overline{E}) + 1.
\]

Altogether, we obtain

\[
z^*_c(\overline{E}) < 2z^*(\overline{E}) + 1
\]

leading to

\[
\Delta(\overline{E}) < \frac{m + 1}{2}
\]

due to Lemma 3 with \( p = 2 \) and \( q = 1 \).

**Remark 10.** One of the main ingredients of the previous proof was the inequality

\[
\frac{l^\top \overline{b}}{L} < 2 \cdot z^*(\overline{E}).
\]
Note that this upper bound is (asymptotically) tight and, therefore, cannot be improved in the general setting. The instances \( E(L) = (1, L - 1, L, 2) \) with \( L \in \mathbb{N}, L \geq 2 \) satisfy \( z^*(E(L)) = 1 \) and

\[
\frac{l^Ta}{L} = 2 \cdot \frac{L - 1}{L} \longrightarrow 2 \cdot z^*(E(L)) \quad \text{as} \quad L \to \infty.
\]

Nevertheless, for most instances, this estimate can be improved if we use some more of the problem-specific input-data. To this end, note that the inequality \( l^Ta < 2L \) for all \( a \in \mathcal{P}^*(E) \) might be very weak in some cases. Instead, it would be better to consider

\[
v_{\max} := v_{\max}(E) := \max \{ l^Ta \mid a \in \mathcal{P}^*(E), a \leq b \}
\]

since this is the greatest total length of a minimal pattern that is used at least once in an integer solution of \( E \). If we apply the FFD heuristic for the dual bin packing problem (see [MS15]) with bin size \( l_{B} = v_{\max} \) to the instance \( E \), then all but possibly the last bin satisfy \( L \leq l_j^a \leq v_{\max} \) for \( j \in \{1, \ldots, q - 1\} \) where \( q \) denotes the number of nonempty bins after the first phase of the algorithm. Similar to the previous proof, we obtain

\[
z^*(E) < \frac{v_{\max}}{L} \cdot z^*(E) + 1.
\]

Then, due to Lemma \( \mathbb{5} \) with \( p = v_{\max}/L \in [1, 2) \) and \( q = 1 \), we can state the following result.

**Theorem 11.** Let \( E = (m, l, L, b) \) be an instance of the skiving stock problem, then

\[
\Delta(E) < m \left( 1 - \frac{L}{v_{\max}} \right) + \frac{L}{v_{\max}}.
\]

Since \( v_{\max} \leq L + l_1 - 1 \leq 2(L - 1) \) holds (for \( L > l_1 > \ldots > l_m \) in \( \mathbb{8} \)), this upper bound is better than the bound provided by Theorem \( \mathbb{9} \). If the item lengths are sufficiently small, we can state the following special case of the previous theorem.

**Corollary 12.** Let \( \tau \in \mathbb{N} \) with \( \tau \geq 2 \) be given. If \( l_i \leq (\tau + 1)/\tau^2 \cdot L \) holds for all \( i \in I \), then the gap satisfies the inequality

\[
\Delta(E) < \frac{m + \tau}{\tau + 1}.
\]

**Proof.** Obviously, it suffices to show that

\[
v_{\max} \leq \frac{\tau + 1}{\tau} \cdot L
\]

since this would prove the assertion by means of \( \mathbb{8} \) and Lemma \( \mathbb{5} \). For the sake of contradiction, we assume that there is a minimal pattern \( a \in \mathbb{Z}^m_+ \) with \( a \leq b \) satisfying \( l^a > (\tau + 1)/\tau \cdot L \). In this case, \( a \in \mathbb{Z}^m_+ \) cannot contain any item of length less than or equal to \( L/\tau \) since, otherwise, \( a \) would not be minimal. Hence, \( a \) consists exclusively of items whose length is greater than \( L/\tau \). In order to be minimal, at most \( \tau \) such objects can be used to build the pattern \( a \). Using the assumption \( l_i \leq (\tau + 1)/\tau^2 \cdot L \), this leads to

\[
\frac{\tau + 1}{\tau} \cdot L < l^a \leq \tau \cdot \frac{\tau + 1}{\tau^2} \cdot L = \frac{\tau + 1}{\tau} \cdot L
\]

which gives the contradiction and completes the proof. \( \square \)
Up to the present, upper bounds related to \( v_{\max}(E) \) appear to be rather non-applicable since the calculation of \( v_{\max} \) would possibly need all minimal patterns to be computed before. Due to the exponential number of patterns, this is mostly neither practicable nor reasonable. Hence, the question arises if (and how) \( v_{\max} \) can be computed (more) efficiently.

In the remainder of this section, we will present a method that is based on solving at most \( m \) bounded knapsack problems which corresponds to a complexity of \( O(m^2L) \), cf. [KPP04, Theorem 7.2.2].

For every \( i \in I \) we define the optimization problem

\[
P(i) : \quad \sum_{k=1}^{i} l_k a_k \to \max \text{ s.t. } \sum_{k=1}^{i} l_k a_k \leq L - 1, a_k \in \mathbb{Z}_+, a_k \leq b_k - \delta_{ik}, 1 \leq k \leq i,
\]

where \( \delta_{ik} \in \{0, 1\} \) denotes the Kronecker delta. Let \( a^*_i \) and \( z^*_i \) denote a solution and the optimal objective value of \( P(i) \), respectively.

**Theorem 13.** Let \( E = (m, l, L, b) \) be an instance of the skiving stock problem with \( l^\top b \geq L \), and let \( z_{\text{max}} \) be defined by

\[
z_{\text{max}} := \max \{ z^*_i + l_i \mid i \in I \}.
\]

Then \( v_{\max} = z_{\text{max}} \) holds.

**Proof.** Due to \( l^\top b \geq L \), \( v_{\max} \) is well-defined and \( v_{\max} \geq L \) holds.

**Part 1:** \( v_{\max} \geq z_{\text{max}} \)

Let \( \mathcal{A} \) be given by

\[
\mathcal{A} := \{ i \in I \mid z^*_i + l_i \geq L \}.
\]

Then, due to \( l^\top b \geq L \), we have \( \mathcal{A} \neq \emptyset \). Hence, we choose an arbitrary but fixed element \( \sigma \in \mathcal{A} \) and consider a corresponding solution \( a^*_{\sigma} \) of \( P(\sigma) \). By adding \( m - \sigma \) zeros, the vector \( a^*_{\sigma} \) can be interpreted as an element of \( \mathbb{Z}^m_+ \). For the sake of simplicity, this vector shall also be referred to as \( a^*_{\sigma} \). In the following, we prove that \( a := a^*_{\sigma} + e_{\sigma} \) represents a minimal pattern with \( a \leq b \) and therefore a feasible point for the optimization problem related to \( v_{\max} \).

At first, we observe that

\[
l^\top a = \sum_{i=1}^{m} l_i a^*_i + l_\sigma = \sum_{i=1}^{\sigma} l_i a^*_i + l_\sigma = z^*_\sigma + l_\sigma \geq L
\]

due to the definition of \( \mathcal{A} \). Consequently, we have \( a \in P(E) \). Additionally, \( a \leq b \) follows immediately from the definition of problem \( P(\sigma) \) and the fact that, afterwards, only one object of length \( l_\sigma \) is added which still is allowed due to the last constraint of \( P(\sigma) \). Thus, only the minimality of \( a \) remains to show. For the sake of contradiction, we assume \( a \in \mathbb{Z}^m_+ \) to be a non-minimal pattern, i.e., there is one object that can be removed from \( a \) while still maintaining \( l^\top a \geq L \). But, due to the construction of \( a \), all of the objects in \( a \) have a length greater than or equal to \( l_\sigma \). Thus, after removing an item, a total length of at most

\[
l^\top a - l_\sigma = l^\top a^*_{\sigma} = z^*_\sigma \leq L - 1
\]
would be left, which gives the contradiction.

Hence, \( a \) is feasible for the maximization (7) and

\[
\begin{align*}
v_{\text{max}} & = \max \{ l^T a \mid a \in P^*(E), a \leq b \} \\
& \geq \max \{ l^T a \mid a \in \{ a^* + e^\tau \mid \tau \in A \} \}
\end{align*}
\]

follows directly.

**Part 2:** \( v_{\text{max}} \leq z_{\text{max}} \)

For the sake of contradiction, we assume that \( v_{\text{max}} > z_{\text{max}} \) holds. Then there exists a minimal pattern \( a \in P^*(E) \) with \( a \leq b \) and \( v_{\text{max}} = l^T a > z_{\text{max}} \). Let

\[
\sigma := \sup \{ j \in \{1, \ldots, m\} \mid a_j \geq 1 \}
\]

be the index of the shortest item appearing in \( a \). Then, due to the minimality of \( a \), the vector \( \bar{a} := a - e^\sigma \) satisfies \( l^T \bar{a} \leq L - 1 \) and \( \bar{a}_\sigma \leq b_\sigma - 1 \). Considering \( \bar{a} \) as an element of \( Z_+^\sigma \), we obtain a feasible solution for the problem \( P(\sigma) \). Consequently, we obtain

\[
l^T a = v_{\text{max}} > z_{\text{max}} \geq z^*_\sigma + l_\sigma \geq l^T \bar{a} + l_\sigma = l^T a = v_{\text{max}}
\]

which gives the contradiction.

Altogether, we have \( v_{\text{max}} \leq z_{\text{max}} \) and \( z_{\text{max}} \leq v_{\text{max}} \) which proves the assertion.

Thus, in order to compute \( v_{\text{max}} \), we have to cope with \( m \) bounded knapsack problems \( P(1), \ldots, P(m) \) in the worst case. Fortunately, there are some observations that might reduce the number of optimization problems to be solved.

- If we know an appropriate lower bound \( \underline{z} \) for \( z_{\text{max}} \), we can define

\[
i_2 := i_2(\underline{z}) := \begin{cases} 
\inf \{ i \in I \mid l_i \leq \underline{z} - (L - 1) \} & \text{if } \{ i \in I \mid l_i \leq \underline{z} - (L - 1) \} \neq \emptyset, \\
\infty & \text{otherwise}.
\end{cases}
\]

Then, problems \( P(\sigma) \) with \( \sigma > i_2 \) do not have to be solved since their optimal objective value \( z^*_\sigma \) cannot satisfy \( z^*_\sigma + l_\sigma > \underline{z} \).

- For instance, such lower bounds can be obtained by the following procedure: let \( i_1 \in \mathbb{N} \) be given by

\[
i_1 := \inf \left\{ i \in I \mid \sum_{k=1}^{i} b_k l_k \geq L \right\}.
\]

Then, the problems \( P(\sigma) \) with \( \sigma < i_1 \) do not have to be solved since their optimal objective value \( z^*_\sigma \) cannot satisfy \( z^*_\sigma + l_\sigma \geq L \). Hence, solving \( P(i_1) \) leads to a possible choice of \( \underline{z} \).
In the special case where \( i_1 = 1 \) holds, obviously, the solution of \( P(1) \) is given by

\[
a^{*1} = \left\lfloor \frac{L - 1}{l_1} \right\rfloor
\]

implying that \( \bar{z} \) can be chosen as

\[
\bar{z} := z^{*1} + l_1 = l_1 \cdot \left( \left\lfloor \frac{L - 1}{l_1} \right\rfloor + 1 \right) = l_1 \cdot \left\lceil \frac{L}{l_1} \right\rceil.
\]

- If we aim at solving the problems \( P(1), \ldots, P(m) \) it is recommendable to start from behind. Due to the second reduction method, the first problem to be considered is \( P(i_2) \). Let \( k \in \{i_1, \ldots, i_2\} \) denote the index of the currently considered problem. If there is a solution \( a^{*k} \) of \( P(k) \) whose last \( l \in \mathbb{N} \) components are all equal to zero, then \( a^{*k} \) (with an appropriately reduced number of components) also describes a solution for the problems \( P(k - 1), \ldots, P(k - l) \). If, additionally,

\[
a_{k-(l+1)}^{*k} \leq b_{k-(l+1)} - 1
\]

holds, it is even a solution of \( P(k - (l + 1)) \). Hence, as the case may be, we can continue with problem \( P(k - (l + 1)) \) or \( P(k - (l + 2)) \) (if \( k - (l + 2) < i_1 \), the calculations are done).

These observations lead to the following algorithm that can be applied to calculate \( v_{\text{max}} \) for a given instance \( E = (m, l, L, b) \) with \( l^\top b \geq L \) and \( L > l_1 > \ldots > l_m \geq 1 \).

**Algorithm 1** Calculation of \( v_{\text{max}} = v_{\text{max}}(E) \)

**Input:** Compute \( i_1 \), solve \( P(i_1) \), and set \( \bar{z} := z^{*1} + l_1 \). Set \( k := i_2(\bar{z}) \).

1: while \( k \geq i_1 \) do
2:  Compute a solution \( a^{*k} \) and the optimal objective value \( z^{*}_k \) of problem \( P(k) \).
3:  if \( z^{*}_k + l_k > \bar{z} \) then
4:      Set \( \bar{z} := z^{*}_k + l_k \).
5:      if \( a^{*k}_k = 0 \) then
6:         Compute \( \sigma = \inf \left\{ j \in \{i_1, \ldots, k - 1\} \mid a^{*k}_i = 0 \text{ for all } i > j \right\} \).
7:         if \( a^{*k}_\sigma \leq b_\sigma - 1 \) then
8:            Set \( k := \sigma - 1 \) and \( \bar{z} := \sum_{i=1}^{\sigma} l_i a^{*k}_i + l_\sigma \)
9:         else
10:            Set \( k := \sigma \) and \( \bar{z} := \sum_{i=1}^{\sigma} l_i a^{*k}_i + l_{\sigma+1} \)
11:       end if
12:      else
13:         \( k := k - 1 \)
14:     end if
15:  end if
16: end while

**Output:** \( \bar{z} = z_{\text{max}} = v_{\text{max}} \)
Two Further Approaches

An obvious possibility to obtain a feasible integer solution from a given solution of the continuous relaxation consists in rounding down each variable $x_j$ ($j \in J^*$) to the nearest integer. This method was also applied in the proof of [MS15, Theorem 1]. In what follows, we aim at evaluating the quality of this approach in terms of the resulting gap. As a starting point, we need the following lemma.

Lemma 14. Let $E = (m, l, L, b)$ be an instance of the skiving stock problem with $l^T b / L \geq \rho - 1$ for some $\rho \in \mathbb{N}$. Then we have $z^*(E) \geq \lfloor \rho/2 \rfloor$.

Proof. Obviously, it suffices to consider $\rho := \lfloor l^T b / L \rfloor + 1$ since this is the maximal choice of $\rho$. Then there are two cases:

i) Case 1: $\rho$ is even
   In this case, we have
   \[ l^T b \geq (\rho - 1)L = (2\tau + 1)L \]
   for some $\tau \in \mathbb{Z}_+$. Since any minimal pattern $a \in P^*(E)$ satisfies $l^T a < 2L$ we, certainly, can build $\tau + 1$ patterns out of the given items. Hence, we have
   \[ z^*(E) \geq \tau + 1 = \frac{\rho + 2}{2} + 1 = \frac{\rho}{2} = \lfloor \frac{\rho}{2} \rfloor \]
   due to $\rho = 2\tau + 2$ and the assumption of this case.

ii) Case 2: $\rho$ is odd
   In this case, we have
   \[ l^T b \geq (\rho - 1)L = 2\tau \cdot L \]
   for some $\tau \in \mathbb{Z}_+$. Since any minimal pattern $a \in P^*(E)$ satisfies $l^T a < 2L$ we, certainly, can build $\tau$ patterns out of the given items. Hence, we have
   \[ z^*(E) \geq \tau = \frac{\rho}{2} = \left\lfloor \frac{\rho}{2} \right\rfloor \]
   due to $\rho = 2\tau + 1$ and the assumption of this case.

\[\Box\]

Theorem 15. Let $E = (m, l, L, b)$ be an instance of the skiving stock problem with $n = |J^*(E)|$, and let $x^e$ denote a solution of the continuous relaxation (2). If

\[ \rho - 1 \leq \sum_{j=1}^{n} (x^e_j - \lfloor x^e_j \rfloor) < \rho \tag{13} \]

holds for some $\rho \in \mathbb{N}$, then

\[ \Delta(E) < \left\lceil \frac{\rho}{2} \right\rceil. \tag{14} \]
Proof. In the case $\rho = 1$, the rounded solution, by definition, always satisfies the IRDP-condition. Hence,

$$\Delta(E) < 1 = \left\lceil \frac{\rho}{2} \right\rceil$$

follows directly. So, let (13) be satisfied for some $\rho \geq 2$. For $i \in I$ we define

$$b_i^* = b_i - \sum_{j=1}^{n} a_{ij} \lfloor x_j^c \rfloor,$$

i.e., the number of items (of type $i \in I$) that still can be used to build further patterns. Due to Lemma 1, we obtain

$$\frac{l^T b^*}{L} = \frac{1}{L} \cdot l^T \left( b - \sum_{j=1}^{n} a^j [x_j^c] \right) = \frac{1}{L} \cdot l^T \left( \sum_{j=1}^{n} (x_j^c - \lfloor x_j^c \rfloor) a^j \right)$$

$$= \frac{1}{L} \cdot \sum_{j=1}^{n} \left( (x_j^c - \lfloor x_j^c \rfloor) \cdot \frac{l^T a^j}{l^T a^j} \geq L \right) \geq \sum_{j=1}^{n} (x_j^c - \lfloor x_j^c \rfloor) \geq \rho - 1.$$

By means of the previous lemma, we have $z^*(E^*) \geq \lfloor \rho/2 \rfloor$ for the residual instance $E^* = (m, l, L, b^*)$ where $b_i^* = 0$ is possible for some $i \in I$. Consequently, we have found a feasible integer solution for the instance $E$ with objective value

$$z \geq \sum_{j=1}^{n} \lfloor x_j^c \rfloor + \left\lfloor \frac{\rho}{2} \right\rfloor$$

implying that

$$\Delta(E) = z^*_c(E) - z^*(E) \leq z_j^*(E) - z \leq \sum_{j=1}^{n} x_j^c - \left( \sum_{j=1}^{n} \lfloor x_j^c \rfloor + \left\lfloor \frac{\rho}{2} \right\rfloor \right)$$

$$= \sum_{j=1}^{n} (x_j^c - \lfloor x_j^c \rfloor) - \left\lfloor \frac{\rho}{2} \right\rfloor \leq \rho - \left\lfloor \frac{\rho}{2} \right\rfloor = \left\lceil \frac{\rho}{2} \right\rceil$$

which proves the assertion.

Obviously, Theorem 15 holds for all instances of the skiving stock problem, particularly for residual ones. This (trivial) observation has an important implication which improves the upper bound of Theorem 9 slightly.

Corollary 16. Let $E = (m, l, L, b)$ denote an instance of the skiving stock problem. Then,

$$\Delta(E) < \left\lceil \frac{m}{2} \right\rceil$$

holds.

Proof. Let $\overline{E} = \overline{E}(x^c)$ be a residual instance of $E$ with $n = |J^*(E)| = |J^*(\overline{E})|$, and let $x^*$ denote a solution of the continuous relaxation of $\overline{E}$. Then we have

$$\sum_{j=1}^{n} x_j^c < m$$
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since $0 \leq \pi < e$ holds (for an appropriate-sized vector $e = (1, \ldots, 1)^\top$) and the basis matrix belonging to the solution $\pi^*$ contains at most $m$ different patterns. Hence, there exists $\rho \in \{1, \ldots, m\}$ such that

$$\rho - 1 \leq \sum_{j=1}^{n} (\pi_j - \lfloor \pi_j \rfloor) < \rho$$

implying that

$$\Delta(E) \leq \Delta(\pi) < \left\lceil \frac{\rho}{2} \right\rceil \leq \left\lceil \frac{m}{2} \right\rceil$$

due to Theorem 15.

In particular, also this approach leads to the result that all instances with $m = 4$ possess the MIRDP. Furthermore, note that the method presented in this proof cannot be improved in the general setting. This is mainly due to the fact, that whenever $l^\top b / L \geq \rho - 1$ holds for some $\rho \in \mathbb{N}$, we cannot obtain a better result than $\lfloor \rho/2 \rfloor$ patterns in the worst case.

One of the main drawbacks of this upper bound is also the fact that it does not use any information about the other given input-data, such as the vectors $l$ and $b$. This means that, for constant $m$, the upper bound is always the same regardless of how the other parameters of $E$ look like. In the remainder of this section we aim at constructing an upper bound that uses as many of the given information as possible. To this end, let $E = (m, l, L, b)$ be an instance of the skiving stock problem. Then we consider $E' = (m, l', L, b)$ where $l'_i$ is given by $l'_i := L / k_i$ with $k_i := \lceil L / l_i \rceil$ for $i \in I$. Note that $E'$ can also be interpreted as an instance, even if we have $l'_i \in \mathbb{Q}^m$, in general.

**Lemma 17.** The inequality $l'_i \leq l_i$ holds for all $i \in I$.

**Proof.** Let $i \in I$ be arbitrarily chosen but fixed. Then we have the equivalence

$$l'_i \leq l_i \iff \frac{L}{k_i} \leq l_i \iff \frac{L}{L / k_i} \leq l_i \iff \frac{l_i}{L} \cdot \left\lceil \frac{L}{l_i} \right\rceil \geq 1$$

and the latter is true due to

$$\frac{l_i}{L} \cdot \left\lceil \frac{L}{l_i} \right\rceil \geq \frac{l_i}{L} \cdot \frac{L}{l_i} \geq 1.$$

**Corollary 18.** The inequalities $z^*(E') \leq z^*(E)$ and $z^*_c(E') \leq z^*_c(E)$ hold.

**Proof.** Due to $l'_i \leq l_i$ for $i \in I$, we have $P(E') \subseteq P(E)$ for the pattern sets of both instances. Hence, any feasible (integer) solution of $E'$ is also a feasible (integer) solution of $E$ which proves the assertion.

Note that $E'$ is an instance of the well-studied divisible case since

$$\frac{L}{l'_i} = \frac{L}{k_i} = k_i \in \mathbb{N}$$
holds for all $i \in I$. Thus, we can state

$$z^*(E') = \sum_{i=1}^{m} \frac{b_i l'_i}{L} = \sum_{i=1}^{m} \frac{b_i \cdot L}{k_i} = \sum_{i=1}^{m} \frac{b_i}{k_i}$$  \hspace{1cm} (16)$$

$$z^*(E') > z^*(E') - 2 = \sum_{i=1}^{m} \frac{b_i}{k_i} - 2$$  \hspace{1cm} (17)

due to the MIRDP of the divisible case, see [MS15]. These preliminaries lead to the following result.

**Theorem 19.** Let $E = (m, l, L, b)$ be an instance of the skiving stock problem, $\bar{E} = E(x^*) = (m, l, \bar{L}, \bar{b})$ one of its residual instances and $E' = (m, l', L, b)$ constructed as above. Then

$$\Delta(E) < 2 + \sum_{i=1}^{m} \frac{\bar{b}_i \cdot (k_i - \frac{L}{l_i})}{k_i \cdot \frac{L}{l_i}} < 2 + \frac{1}{2} \sum_{i=1}^{m} \frac{\bar{b}_i l_i}{L}$$

holds.

**Proof.** We have

$$\Delta(E) \leq \Delta(\bar{E}) = z^*_c(\bar{E}) - z^*(\bar{E}) \leq \sum_{i=1}^{m} \frac{\bar{b}_i l_i}{L} - z^*(\bar{E})$$

$$\leq \sum_{i=1}^{m} \frac{\bar{b}_i l_i}{L} - z^*(E') < \sum_{i=1}^{m} \frac{\bar{b}_i l_i}{L} - \sum_{i=1}^{m} \frac{b_i}{k_i} + 2$$

$$= 2 + \sum_{i=1}^{m} \frac{\bar{b}_i \cdot (k_i - \frac{L}{l_i})}{k_i \cdot \frac{L}{l_i}}$$

which proves the first inequality. Then, due to $k_i \geq 2$ for all $i \in I$, observe that

$$2 \leq k_i < \frac{L}{l_i} + 1$$

holds which leads to

$$\frac{2}{k_i} \leq 1 < \frac{L/l_i}{k_i} + \frac{1}{k_i} \implies \frac{L/l_i}{k_i} > 1 - \frac{1}{k_i} \geq \frac{1}{2}.$$  

Now, we have

$$2 + \sum_{i=1}^{m} \frac{\bar{b}_i \cdot (k_i - \frac{L}{l_i})}{k_i \cdot \frac{L}{l_i}} = 2 + \sum_{i=1}^{m} \bar{b}_i \left( \frac{l_i}{L} - \frac{1}{k_i} \right) = 2 + \sum_{i=1}^{m} \frac{\bar{b}_i l_i}{L} \left( 1 - \frac{L/l_i}{k_i} \right)$$

$$< 2 + \frac{1}{2} \sum_{i=1}^{m} \frac{\bar{b}_i l_i}{L}$$

and the proof is complete. \qed
In [Rie03], a similar approach for the gap of the cutting stock problem is presented, but there the obtained upper bound is of worse quality due to some weaker estimation within the derivation. In our case, there are still at least two main possibilities to improve the upper bound:

- The constant term of the upper bound can be improved if it is possible to find a better upper bound for the gap of the divisible case. In the context of one-dimensional cutting, an upper bound of $\Delta(E) < 1.4$ has been proved for the divisible case, cf. [Rie03]. Up to now, it is not known whether a similar result can be shown for the skiving stock problem, too. Note that in the special case where $E'$ consists of only two item types, i.e., $|\{l'_1, \ldots, l'_m\}| = 2$, the summand 2 can be replaced by 1 due to $\Delta(E) < \max\{1, m-1\}$.

- A very weak inequality, used in the previous estimate, is $z_c^*(E) \leq \sum_{i=1}^{m} \frac{\overline{b}_i l_i}{L}$ since the absolute difference of both terms can be arbitrarily large. Thus, the upper bound can be improved significantly if, for instance, a constant $C \in \mathbb{R}_+$ with $z_c^*(E) \leq z_c^*(E') + C$ could be found.

Nevertheless, Theorem 19 provides an upper bound that uses all of the input-data and improves the bound given by Theorem 15 in some cases, particularly for large values of $m$ and small values of $b_i$ ($i \in I$). The latter can be seen by

$$\Delta(E) < 2 + \sum_{i=1}^{m} \frac{\overline{b}_i \cdot \left(k_i - \frac{L}{l_i}\right)}{k_i \cdot \frac{L}{l_i}} < 2 + \sum_{i=1}^{m} \frac{\overline{b}_i}{k_i \cdot \frac{L}{l_i}} \leq 2 + \max\left\{\overline{b}_i \left| i \in \{1, \ldots, m^*\}\right.\right\} \cdot \sum_{i=1}^{m^*+1} \frac{1}{k_i^* \cdot (k_i^* - 1)}$$

$$\leq 2 + \max\left\{\overline{b}_i \left| i \in \{1, \ldots, m^*\}\right.\right\} \cdot \sum_{i=2}^{m^*+1} \frac{1}{i(i-1)}$$

$$= 2 + \max\left\{\overline{b}_i \left| i \in \{1, \ldots, m^*\}\right.\right\} \cdot \sum_{i=2}^{m^*+1} \left(\frac{1}{i-1} - \frac{1}{i}\right)$$

$$= 2 + \max\left\{\overline{b}_i \left| i \in \{1, \ldots, m^*\}\right.\right\} \cdot \left(1 - \frac{1}{m^*+1}\right),$$

where $m^*$ and $\overline{b}_i^*$ ($i \in I$) are given by

$$m^* := |\{l'_1, \ldots, l'_m\}| \quad \text{and} \quad \overline{b}_i^* := \sum_{j \in I, l'_i = l'_j} \overline{b}_j,$$

and $k_i^*$ (defined as usual) belongs to one of the $m^*$ different item lengths.
5 Conclusion and outlook

In this paper, we investigated the gap of the skiving stock problem from a theoretical point of view. As a starting point, we introduced the theory of residual instances and significantly simplified the proof of [MS15, Theorem 1]. Afterwards, we stated several new and improved upper bounds for the gap of (general) instances of the skiving stock problem, also showing that any instance with \( m = 4 \) possesses the MIRDP. Additionally, we presented an approach to compute an upper bound by considering a related instance that belongs to the well-studied divisible case, see also [MS15].

In the context of one-dimensional cutting, estimates like \( \Delta_{\text{CSP}}(E) < \max\{2, (m + 2)/4\} \) (see [Ric03, p.61]) are already available. Therefore, we aim at improving the upper bound \((m + 1)/2\) (or \(\lceil m/2 \rceil\), respectively) introduced in this paper in order to get a bit closer to the MIRDP-conjecture. Another main objective is given by the consideration of stronger relaxations, like the proper relaxation, and investigations related to the behaviour of the corresponding (proper) gaps.

References


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