Integer Rounding and Modified Integer Rounding for the Skiving Stock Problem

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Integer Rounding and Modified Integer Rounding for the Skiving Stock Problem

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Abstract

We consider the one-dimensional skiving stock problem which is also known as the dual bin packing problem: find the maximum number of items with minimum length $L$ that can be constructed by connecting a given supply of $m \in \mathbb{N}$ smaller item lengths $l_1, \ldots, l_m$ with availabilities $b_1, \ldots, b_m$. For this optimization problem, we investigate the quality of the continuous relaxation by considering the gap, i.e., the difference between the optimal objective values of the continuous relaxation and the skiving stock problem itself. In a first step, we derive an upper bound for the gap by generalizing a result of E. J. Zak. As a main contribution, we prove the modified integer round-down property of the divisible case. In this context, we also present a construction principle for non-IRDP instances of the divisible case and investigate the maximum gap that can be obtained based on these instances.

Key words: Skiving Stock Problem, (Modified) Integer Round Down Property, Gap, Divisible Case, Continuous Relaxation

1 Introduction and Preliminaries

In this paper, we consider the one-dimensional skiving stock problem (SSP) which is also known as the dual bin packing problem (DBPP) in literature. In the classical formulation, $m \in \mathbb{N} := \{1, 2, \ldots\}$ different item lengths $l_1, \ldots, l_m$ with availabilities $b_1, \ldots, b_m$ are given, the so-called item supply. We aim at maximizing the number of products with minimum length $L$ that can be constructed by connecting the items on hand.

Such computations are of high interest in many real world applications, e.g. industrial production processes (see [Zak03] for an overview) or politico-economic problems (cf. [AJKL84] and [LLM95]). Furthermore, also neighboring tasks, such as dual vector packing problem [CFGR91] or the maximum cardinality bin packing problem [BD85], [PD06], are often associated or even identified with the dual bin packing problem. These formulations are of practical use as well since they are applied in multiprocessor scheduling problems [ARGA04] or surgical case plannings [VPS+13].
Throughout this paper, we will use the abbreviation $E := (m, l, L, b)$ for an instance of the SSP with $l = (l_1, \ldots, l_m)^\top$ and $b = (b_1, \ldots, b_m)^\top$. Without loss of generality, we assume all input-data to be positive integers with $L > l_1 > \ldots > l_m > 0$.

The classical solution approach is due to \cite{Zak03} and based on the formulation of Gilmore and Gomory in the context of one-dimensional cutting \cite{GG61}. Any feasible arrangement of items leading to a final product of minimum length $L$ is called \textit{(packing) pattern} of $E$. We always represent a pattern by a nonnegative vector $a = (a_1, \ldots, a_m)^\top \in \mathbb{Z}_+^m$ where $a_i \in \mathbb{Z}_+$ denotes the number of items of type $i \in I := \{1, \ldots, m\}$ being contained in the considered pattern. For a given instance $E$, the set of all patterns is defined by $P_E := \{a \in \mathbb{Z}_+^m \mid \sum_i a_i \geq L\}$. In \cite{MS14}, the authors slightly improve Zak’s formulation by only considering so-called minimal patterns obtaining a finite model, hereinafter referred to as the \textit{standard model}, of the skiving stock problem. A pattern $a \in P_E$ is called \textit{minimal} if there exists no pattern $\bar{a} \in P_E$ such that $\bar{a} \neq a$ and $\bar{a} \leq a$ hold (componentwise). The set of all minimal patterns is denoted by $\tilde{P}_E$. Let $x_j \in \mathbb{Z}_+$ denote the number how often the minimal pattern $a^j = (a_{1j}, \ldots, a_{mj})^\top \in \mathbb{Z}_+^m \ (j \in J)$ of $E$ is used, where $J = \{1, \ldots, n\}$ represents an index set of all minimal patterns. Then the skiving stock problem can be formulated as

$$z^*(E) = \max \left\{ \sum_{j \in J} x_j \ \bigg| \ \sum_{j \in J} a_{ij} x_j \leq b_i, \ i \in I, \ x_j \in \mathbb{Z}_+, \ j \in J \right\}.$$ 

A common (approximate) solution approach consists in considering the continuous relaxation

$$z^c_\epsilon(E) = \max \left\{ \sum_{j \in J} x_j \ \bigg| \ \sum_{j \in J} a_{ij} x_j \leq b_i, \ i \in I, \ x_j \geq 0, \ j \in J \right\}$$

and the application of appropriate heuristics.

Practical experience and computational simulations, cf. \cite{Zak03}, have shown that there is only a small gap $\Delta(E) := z^c_\epsilon(E) - z^*(E)$ for any instance $E$. Based on the contributions of Baum and Trotter \cite{BT81} for general linear maximization problems, these observations have initiated the following definitions. A set $\mathcal{P}$ of instances has the \textit{integer round-down property} (IRDP) if $\Delta(E) < 1$ holds for all $E \in \mathcal{P}$. An instance $E$ with $\Delta(E) \geq 1$ is called \textit{non-IRDP instance}. Furthermore, a set $\mathcal{P}$ of instances has the \textit{modified integer round-down property} (MIRDP) if $\Delta(E) < 2$ is true for all $E \in \mathcal{P}$. It is conjectured in \cite{Zak03} that the one-dimensional skiving stock problem possesses the MIRDP.

In this paper, we investigate the gap of the skiving stock problem from a theoretical point of view. To this end, we first take the result of \cite[Theorem 4]{Zak03}, i.e., the proof of the IRDP for all instances with $m = 2$ item lengths, as an initial point and generalize this inequality to arbitrary values of $m$. This case and other special cases have also been studied by Marcotte \cite{Mar83} in the context of one-dimensional cutting.

In Section 3, we focus intensively on the \textit{divisible case} of the skiving stock problem where $l_i | L$ holds for each item length $l_i \ (i \in I)$. For these instances, we prove the MIRDP by means of the \textit{first fit decreasing} (FFD) heuristic presented in \cite{AJKL84} for the dual bin packing problem. Afterwards, we introduce a construction principle for an infinite number of non-equivalent non-IRDP instances of the divisible case. In a final step, we investigate the maximum gap that can be obtained on the basis of these instances, give some conclusions and an outlook of future research.
2 A generalization of Zak’s theorem

To our best knowledge, there is only few work concerning theoretical aspects of the skiving stock problem’s gap. One of the only results being known in literature is Theorem 4 in [Zak03] where the IRDP for instances with $m = 2$ item lengths is proved. In the following theorem, we generalize this statement such that it provides an upper bound for the gap of arbitrary instances $E = (m, l, L, b)$.

**Theorem 1.** Let $m \in \mathbb{N}$ with $m \geq 2$ be given and let $E = (m, l, L, b)$ be an arbitrary instance with $m$ item lengths. Then $\Delta(E) < m - 1$ holds.

**Proof.** Let $n := |J|$ denote the number of all minimal patterns of $E$. Then, consider the optimization problem

$$
c^\top x \rightarrow \text{max} \quad \text{s.t.} \quad (A \ I) x = b, \ x \geq 0 \quad (1)
$$

where $c \in \mathbb{Z}^{n+m}_+$ is defined by $c_j = 1$ for $j \in \{1, \ldots, n\}$ and $c_j = 0$ for $j \geq n + 1$. Note that the columns of $A$ consist of the minimal patterns of $E$, and $I$ represents the appropriate identity matrix. Let $j_1, \ldots, j_m \in \{1, \ldots, n + m\}$ denote the (pairwise different) indices of those columns of $(A \ I)$ that belong to the basis matrix of an arbitrary but fixed solution $x^{LP} \in \mathbb{R}^{n+m}_+$. Thus, we have $x^{LP}_j = 0$ for all $j \in \{1, \ldots, n + m\} \setminus \{j_1, \ldots, j_m\}$.

i) **Case 1:** There is an index $j_k \in \{j_1, \ldots, j_m\}$ with $j_k \geq n + 1$.

Let $A \subseteq \{j_1, \ldots, j_m\}$ contain all these indices. Then, we obtain an integer solution by means of

$$
x_j := \begin{cases} 
\lfloor x^{LP}_j \rfloor & \text{for } j \in \{j_1, \ldots, j_m\} \setminus A, \\
0 & \text{otherwise.}
\end{cases}
$$

Due to $|A| \geq 1$,

$$
z^*_c(E) - z^*(E) \leq \sum_{j=1}^{n+m} x^{LP}_j - \sum_{j=1}^{n+m} x_j
$$

$$
= \sum_{j \in \{j_1, \ldots, j_m\} \setminus A} \left( x^{LP}_j - x_j \right)
$$

$$
< |\{j_1, \ldots, j_m\} \setminus A| = m - |A| \leq m - 1
$$

follows which proves the assertion.

ii) **Case 2:** $j_k \leq n$ holds for all $j_k \in \{j_1, \ldots, j_m\}$.

Without loss of generality, we set $j_1 = 1, \ldots, j_m = m$. Then, the equality

$$
\sum_{j=1}^{m} a_{ij} x^{LP}_j = b_i \quad (2)
$$

holds for all $i \in \{1, \ldots, m\}$. If there were an index $k \in \{1, \ldots, m\}$ with $x^{LP}_k \in \mathbb{Z}_+$, we would obtain an integer solution by means of

$$
x_j := \begin{cases} 
\lfloor x^{LP}_j \rfloor & \text{for } j \in \{1, \ldots, m\}, \\
0 & \text{otherwise}
\end{cases}
$$
proving the assertion due to
\[
z^*_c(E) - z^*(E) \leq \sum_{j=1}^{n+m} x_j^{LP} - \sum_{j=1}^{n+m} x_j = \sum_{j=1}^{m} (x_j^{LP} - x_j)
= \sum_{j=1}^{m} (x_j^{LP} - x_j) + (x_k^{LP} - x_k) = \sum_{j=1}^{m} (x_j^{LP} - x_j) < m - 1.
\]

Thus, only the case where \(x_j^{LP}\) (for all \(j \in \{1, \ldots, m\}\)) is not an integer remains. To this end, we consider the \(m\)-dimensional hypercube \(W(m)\) (see Fig. 1) described by
\[
W(m) = \{ y \in \mathbb{R}_+^m \mid \forall j \in \{1, \ldots, m\} : y_j \in [\lfloor x_j^{LP} \rfloor, \lceil x_j^{LP} \rceil] \}.
\]

![Figure 1: The cube \(W(2)\) with the areas for the subsequent case-by-case analysis. Triangle ABD (without the segment BD) corresponds to Case 2.1, triangle BCD corresponds to Case 2.2.](image)

Due to the non-integrality assumption, the projection of \(x^{LP}\) onto the first \(m\) components belongs to the interior of \(W(m)\).

a) **Case 2.1:** Let
\[
\sum_{j=1}^{m} (x_j^{LP} - \lfloor x_j^{LP} \rfloor) < 1.
\]

Then,
\[
\sum_{j=1}^{m} (x_j^{LP} - \lfloor x_j^{LP} \rfloor) < 1
\iff
\sum_{j=1}^{m} x_j^{LP} - \sum_{j=1}^{m} \lfloor x_j^{LP} \rfloor < 1
\iff
z^*_c(E) - z^*(E) \leq \sum_{j=1}^{m} x_j^{LP} - \sum_{j=1}^{m} \lfloor x_j^{LP} \rfloor < 1 \leq m - 1
\]
follows since
\[
x_j := \begin{cases} 
\lfloor x_j^{LP} \rfloor & \text{for } j \in \{1, \ldots, m\}, \\
0 & \text{otherwise}
\end{cases}
\]
represents an integer solution. Thus, the assertion has been proved for Case 2.1.

b) **Case 2.2:** Let
\[ \sum_{j=1}^{m} \left( x_{j}^{LP} - \lfloor x_{j}^{LP} \rfloor \right) \geq 1. \]

Defining
\[ \Delta x_{j} := x_{j}^{LP} - \lfloor x_{j}^{LP} \rfloor > 0 \]
for all \( j \in \{1, \ldots, m\} \) and
\[ a_{i}^{\ast} := b_{i} - \sum_{j=1}^{m} a_{ij} \lfloor x_{j}^{LP} \rfloor \]
for all \( i \in \{1, \ldots, m\} \), we obtain, by means of (2),
\[ \forall i \in \{1, \ldots, m\} : a_{i}^{\ast} = \sum_{j=1}^{m} a_{ij} \Delta x_{j}. \]

Then, \( a^{\ast} := (a_{1}^{\ast}, \ldots, a_{m}^{\ast})^{\top} \in \mathbb{Z}_{+}^{m} \) describes a packing pattern since we obtain
\[ l^{\top} a^{\ast} = \sum_{i=1}^{m} l_{i} a_{i}^{\ast} = \sum_{i=1}^{m} l_{i} \sum_{j=1}^{m} a_{ij} \Delta x_{j} \]
\[ = \sum_{j=1}^{m} \Delta x_{j} \sum_{i=1}^{m} l_{i} a_{ij} = \sum_{j=1}^{m} \Delta x_{j} \cdot (l^{\top} a^{\ast}) \]
\[ \geq \sum_{j=1}^{m} \Delta x_{j} \cdot L = L \cdot \sum_{j=1}^{m} \Delta x_{j} \geq L \]
with the help of this case’s assumption.

Without loss of generality, we assume \( a^{\ast} \) to be a minimal pattern. Otherwise, some entries can be repeatedly reduced until this situation is achieved. Thus, \( a^{\ast} \) has to be a column of \( A \). If \( a^{\ast} \) does not match to one of the patterns \( a^{1}, \ldots, a^{m} \), we can set \( a^{m+1} = a^{\ast} \) obtaining an integer solution by means of
\[ x_{j} := \begin{cases} \lfloor x_{j}^{LP} \rfloor & \text{for } j \in \{1, \ldots, m\}, \\ 1 & \text{for } j = m + 1, \\ 0 & \text{otherwise} \end{cases} \]

since
\[ [(A \ I) x]_{i} = \sum_{j=1}^{m} a_{ij} \lfloor x_{j}^{LP} \rfloor + \sum_{j=1}^{m} a_{ij} \Delta x_{j} = \sum_{j=1}^{m} a_{ij} (\lfloor x_{j}^{LP} \rfloor + \Delta x_{j}) \]
\[ = \sum_{j=1}^{m} a_{ij} x_{j}^{LP} = b_{i} \]
holds for all \( i \in \{1, \ldots, m\} \). Otherwise, there exists \( k \in \{1, \ldots, m\} \) with \( a^k = a^* \).

In this case, we define

\[
x_j := \begin{cases} 
\lfloor x_j^{LP} \rfloor & \text{for } j \in \{1, \ldots, m\} \setminus \{k\}, \\
\lfloor x_j^{LP} \rfloor + 1 & \text{for } j = k, \\
0 & \text{otherwise}
\end{cases}
\]

also representing an integer solution since

\[
\begin{align*}
\left( A I \right)_i &= \sum_{j=1, \ j \neq k}^m a_{ij} \lfloor x_j^{LP} \rfloor + a_{ik} \cdot \left( \lfloor x_k^{LP} \rfloor + 1 \right) = \sum_{j=1}^m a_{ij} \lfloor x_j^{LP} \rfloor + a_{ik} \\
\left( a_{ik}=a^*_i \right) &= \sum_{j=1}^m a_{ij} \lfloor x_j^{LP} \rfloor + \sum_{j=1}^m a_{ij} \Delta x_j = \sum_{j=1}^m a_{ij} \left( \lfloor x_j^{LP} \rfloor + \Delta x_j \right) \\
&= \sum_{j=1}^m a_{ij} x_j^{LP} = b_i
\end{align*}
\]

is true for all \( i \in \{1, \ldots, m\} \).

In both cases, we obtain an objective value of \( z = \sum_{j=1}^m x_j^{LP} + 1 \) implying that

\[
z_c^*(E) - z^*(E) \leq \sum_{j=1}^m x_j^{LP} - \left( \sum_{j=1}^m \lfloor x_j^{LP} \rfloor + 1 \right) = \sum_{j=1}^m (x_j^{LP} - \lfloor x_j^{LP} \rfloor) - 1 < m - 1.
\]

Thus, the assertion has been proved for Case 2.2.

\[ \square \]

This theorem provides first insights on instances having the IRDP and MIRDP, respectively. In particular, Theorem 4 from [Zak03] is contained as a special case.

**Corollary 2.** An instance \( E = (m, l, L, b) \) of the skiving stock problem with

i) \( m = 2 \) has the IRDP.

ii) \( m = 3 \) has the MIRDP.

**Remark 3.** Note that instances with \( m = 3 \) item lengths, in general, do not possess the IRDP. Considering \( E = (m, l, L, b) = (3, (21, 14, 6), 42, (1, 2, 6)) \), we notice that

\[
\Delta(E) = z_c^*(E) - z^*(E) = \frac{85}{42} - 1 = \frac{43}{42} > 1.
\]

Thus, \( E \) is a non-IRDP instance.
3 The MIRDP of the divisible case

In this section, we consider a special class of instances of the skiving stock problem. If an instance $E = (m, l, L, b)$ satisfies $l_i | L$ for all $i \in I$, then $E$ is an instance of the divisible case. In the context of one-dimensional cutting, those instances have turned out to possess the modified integer round-up property (MIRUP), see [ST92] or [Rie03, page 24ff.]. In the following, we prove that such a result can be obtained for the skiving stock problem, too. Thereby, the a priori knowledge of the continuous relaxation’s optimal objective value has been proved beneficial.

**Lemma 4.** Let $E = (m, l, L, b)$ be an instance of the divisible case. Then

$$z^*(E) = \frac{l^\top b}{L}$$

(3)

holds.

**Proof.** Let $k_i := L/l_i$ be defined for all $i \in I$ and let $e^i \in \mathbb{Z}_m^+$ denote the $i$-th unit vector. Considering the patterns $a^i = k_i \cdot e^i$ and applying them $x_i = b_i/k_i$ times ($i \in I$), we obtain a feasible solution since

$$\sum_{j=1}^{m} a_{ij} x_j = k_i \cdot x_i = k_i \cdot \frac{b_i}{k_i} = b_i$$

holds for all $i \in I$. Moreover, the objective value results to

$$z = \sum_{j=1}^{m} x_j = \sum_{j=1}^{m} \frac{b_j}{k_i} = \sum_{i=1}^{m} \frac{b_i l_i}{L} = \frac{l^\top b}{L}.$$ 

Thus, we have found a feasible solution whose objective value equals the upper bound $l^\top b/L$ proving the optimality of the given feasible solution. \qed

Note that the patterns $a^i$ ($i \in I$) from the proof satisfy $l^\top a^i = L$. Those patterns are called **exact patterns**.

For instances $E = (m, l, L, b)$ of the divisible case, we firstly apply a scaling of the involved lengths such that $L$ is set to $L' = 1$, and the (modified) item lengths belong to the set $\{1/2, 1/3, 1/4, \ldots\}$. Then, the resulting instance is called **standard form** of $E$.

**Lemma 5.** Let $E = (m, l, L, b)$ be an instance of the divisible case. Then, $E$ is equivalent to $\tilde{E} = (m, \kappa, 1, b)$ where the entries of $\kappa = (1/k_1, \ldots, 1/k_m)^\top$ are given by $k_i := L/l_i$ for all $i \in I$.

**Proof.** It suffices to show that $E$ and $\tilde{E}$ possess the same patterns:

- $a$ is a pattern of $E$
  \[ \iff l^\top a \geq L, \ a \in \mathbb{Z}_+^m \]
  \[ \iff \frac{1}{L} l^\top a \geq 1, \ a \in \mathbb{Z}_+^m \]
  \[ \iff \left( \frac{1}{L} \right)^\top a \geq 1 \]
  \[ = \kappa^\top a \]
  \[ \iff a \ is a \ pattern \ of \ \tilde{E} \]
Hence, both corresponding optimization problems are equivalent.

Let $E = (m, l, L, b)$ be an instance of the divisible case in standard form, i.e., $L = 1$. At first, we only consider the case where $E$ does not contain any exact pattern $a \in P^*_E$ with $a \leq b$. Thus, in particular,

\[ l_i \cdot b_i < 1 \quad \text{for all } i = 1, \ldots, m \]

holds. Now, the first fit decreasing (FFD) algorithm, proposed in [AJKL84] (cf. Algorithm 1) for the dual bin packing problem, represents an initial point of our further investigations. Thereby, bins of capacity $l_B = L = 1$ are filled successively. A bin $B$ is called filled if the objects allocated to it possess a total length of at least $l_B$. If this total length equals $l_B$, the bin $B$ is said to be filled exactly. Furthermore, we say that a bin $B$ can accept an object if, after adding this item, the total length of all objects in $B$ does not exceed $l_B$.

**Algorithm 1 First Fit Decreasing**

**Phase I:**

i) Let $\tilde{n} = e^T b$, with $e = (1, \ldots, 1)^T \in \mathbb{Z}^m_+$, be the total number of items. Sort them according to decreasing lengths and number them consecutively:

\[ 1 > l_1^* \geq l_2^* \geq \ldots \geq l_{\tilde{n}}^*. \]

ii) If there are unallocated objects: Let $i \in \{1, \ldots, \tilde{n}\}$ be the index of the first unallocated object. Choose the first bin that is able to accept this object and place it therein. If such a bin does not exist, add a new (empty) bin and place the object therein.

**Phase II:**

i) If there is more than one nonempty bin that is not filled: consider the last of these bins, choose one of its items and allocate it to the first of these bins.

ii) If there is exactly one nonempty bin that is not filled: allocate its objects to the last bin that is filled.

Note that after Phase II each non-empty bin contains objects with a total length of at least $L = 1$. Hence, each bin can be interpreted as a uniquely determined packing pattern of $E$. Actually, observe that all bins except for the very last one correspond to a minimal pattern. However, also the last non-empty bin can easily be reduced to a minimal pattern.

**Lemma 6.** Let $E = (m, l, 1, b)$ be an instance of the divisible case in standard form not possessing any exact pattern $a \in P^*_E$ with $a \leq b$. Furthermore, let $q \in \mathbb{N}$ denote the number of existing bins after Phase I of Algorithm 1. Then, for all $j \in \{1, \ldots, q - 1\}$, the $j$-th bin $B_j$ contains at least $j + 1$ objects after the completion of Phase I.

**Proof.** Let $j \in \{1, \ldots, q - 1\}$ be given and let $l_j^i \in \{l_1, \ldots, l_m\}$ denote an object length that is contained in bin $B_j$. Then, we claim that

\[ l_j^i \leq \frac{1}{j + 1} \]
holds: Due to (4), Phase I of Algorithm 1 needs at most \( j - 1 \) bins for the allocation of all objects with \( l_i^* \geq 1/j \). Hence, all objects with lengths \( l_i \geq 1/(j+1) \) are contained in the bins \( B_1, \ldots, B_{j-1} \) implying that the \( j \)-th bin \( B_j \) only possesses objects with lengths \( \leq 1/(j+1) \). Since we have \( j < q \), there exists a bin \( B_{j+1} \). Based on the same considerations, all of its objects' lengths are \( \leq 1/(j+2) \). Let \( l^* \) be the largest of these objects lengths, i.e., \( l^* \leq 1/(j+2) \) holds particularly. Due to Phase I of Algorithm 1 the remaining capacity \( R(j) \) of \( B_j \) results to

\[
R(j) < l^* \leq \frac{1}{j+2}
\]

which gives the contradiction, see (6). Thus, our assumption was wrong and the lemma has been proved.

The following lemma states a lower bound for the number \( q \) of bins that exist after Phase I of Algorithm 1. Thereby, the assumption that \( E \) does not contain any exact pattern \( a \in P^*_E \) with \( a \leq b \) plays an important role.

**Lemma 7.** Let \( E = (m, l, 1, b) \) be an instance of the divisible case in standard form. If \( E \) does not contain any exact pattern \( a \in P^*_E \) with \( a \leq b \) and satisfies

\[
\lfloor l^T b \rfloor = K
\]

for some \( K \in \mathbb{N} \), then the inequality \( q \geq K + 1 \) holds.

**Proof.** We assume that there were at most \( K \) bins after the completion of Phase I. Then, the total length of all objects would be strictly smaller than \( K \) since no bin is filled exactly. This gives the contradiction.

Now we have all necessary preliminaries to prove our main result.

**Theorem 8.** Let \( E \) be an instance of the divisible case in standard form not containing any exact pattern \( a \in P^*_E \) with \( a \leq b \). Then \( E \) has the MIRDP.

**Proof.** Let \( q \) denote the number of existing bins after Phase I. Then, certainly, \( q \geq \lfloor l^T b \rfloor + 1 \) holds. Let \( a_{ij} \) be the number of items of type \( i \in \{1, \ldots, m\} \) that have been assigned to the bin \( j \in \{1, \ldots, q\} \). Due to the assumption of this theorem,

\[
\sum_{i=1}^{m} l_i a_{ij} < 1
\]

holds for all \( j \in \{1, \ldots, q\} \).
Let $l^q_i \in \{l_1, \ldots, l_m\}$ be the length of an arbitrary object of bin $B_q$. Then, we obtain

$$\forall j \in \{1, \ldots, q-1\} : 1 - \sum_{i=1}^{m} l_ia_{ij} < l^q_i$$  \hspace{1cm} (7)$$

since, otherwise, the considered object would have been assigned to a previous bin.

i) **Case 1:** $B_q$ contains at least $q - 1$ objects.

Due to (7), each one of these $q - 1$ objects can be assigned to one of the bins $B_1, \ldots, B_{q-1}$ leading to a total length of at least $L = l_B$ in each of these bins. Since

$$q \geq \lceil l^Tb \rceil + 1$$

holds, we found a feasible solution for the ILP with objective value $z = q - 1 \geq \lceil l^Tb \rceil$. In this case, we can even state $z^*(E) = \lceil l^Tb \rceil$ leading to the IRDP of $E$ which implies the MIRDP, too.

ii) **Case 2:** $B_q$ contains $k < q - 1$ objects.

By means of the same procedure as in the first case, the bins $B_1, \ldots, B_k$ can be filled. Now, we consider the bin $B_{q-1}$. Let $l^{q-1}_i$ be an arbitrary item length that is contained in $B_{q-1}$. Then, in analogy to (7), we obtain

$$1 - \sum_{i=1}^{m} l_ia_{ij} < l^{q-1}_i$$

for all $j = k + 1, \ldots, q - 2$ since, otherwise, this object would have been assigned to a previous bin by Phase I of the algorithm. Due to Lemma 6, $B_{q-1}$ contains at least $q$ objects. As described in the first case, they can be used to fill the bins $B_{k+1}, \ldots, B_{q-2}$. Since

$$q \geq \lceil l^Tb \rceil + 1$$

holds, we found a feasible solution for the ILP with objective value $z = q - 2 \geq \lceil l^Tb \rceil - 1$. Hence, we obtain $z^*(E) \geq \lceil l^Tb \rceil - 1$ proving the MIRDP.

The previous investigations could only cope with special instances of the divisible case since we required the absence of exact patterns with $a \leq b$ for our argumentations. In the following, we show how Theorem 8 can be applied to arbitrary instances of the divisible case.

**Theorem 9.** Let $E = (m, l, 1, b)$ be an instance of the divisible case in standard form. Then $E$ has the MIRDP.

**Proof.** We divide $E$ into two subinstances $E_1 = (m, l, 1, \alpha)$ and $E_2 = (m, l, 1, \beta)$. Therefore, we remove successively exact patterns $a \in P_E$ with $a \leq b$ from $E$ and assign their corresponding items to $E_2$. The remaining instance, without exact patterns satisfying $a \leq b$, is called $E_1$. Note that this partition is not unique. For the sake of simplicity, we allow $\alpha_i$ and $\beta_i$ to equal zero for some $i \in \{1, \ldots, m\}$ thus obtaining $m$ item types in both subinstances.
i) subinstance $E_1$:
Here, we can apply Theorem 8 since $E_1$ matches all of its prerequisites, and obtain $z_1 \geq \lceil l^\top \alpha \rceil - 1$.

ii) subinstance $E_2$:
Since $E_2$ consists of (disjoint) exact patterns, we have $\lceil l^\top \beta \rceil = l^\top \beta$ leading to $z_2 = l^\top \beta$.

Since $E_1$ and $E_2$ share no common objects, a feasible solution of $E$ can be obtained by combining the subinstances’ feasible solutions. This results to an objective value of

$$z = z_1 + z_2 
\geq (\lceil l^\top \alpha \rceil - 1) + l^\top \beta 
\overset{(l^\top \beta \in \mathbb{Z}_+) \text{ or } (\alpha + \beta = b)}{=} \lceil l^\top \alpha + l^\top \beta \rceil - 1 
\overset{(\alpha + \beta = b)}{=} \lceil l^\top b \rceil - 1,$$

i.e., the optimal objective values satisfies $z^*(E) \geq \lceil l^\top b \rceil - 1$ proving the MIRDP.

Obviously, this result cannot be improved since there are instances of the divisible case not possessing the IRDP, cf. Remark 3.

4 A construction principle for gaps greater than one

In the previous explanations, we proved the MIRDP for the divisible case, but only indicated a single example of a non-IRDP instance. Thus, the question arises whether there are many of those instances at all. In this section, we present a construction principle for an infinite number of non-equivalent non-IRDP instances and calculate their gap explicitly.

As an initial point, we state the following necessary condition.

Lemma 10. Let $E = (m, l, 1, b)$ be an instance of the divisible case in standard form. If $\Delta(E) \geq 1$ holds, then

$$K := K(E) := \lceil l^\top b \rceil \geq 2$$

is true.

Proof. Assuming that the claim is wrong, there are two cases left: If $K = 0$ holds, the optimal objective value of the ILP equals zero thus obtaining a contradiction by

$$\Delta(E) = z^*_c(E) - z^*(E) = l^\top b - 0 = l^\top b < 1.$$ 

If, in the other case, $K = 1$ holds, the optimal objective value of the ILP equals one thus obtaining a contradiction by

$$\Delta(E) = z^*_c(E) - z^*(E) = l^\top b - 1 < 2 - 1 = 1.$$ 

Hence, the assumption was wrong and the lemma is proved.
Consequently, we can restrict our investigations to instances with $K \geq 2$. So, let $K \in \mathbb{N}$ with $K \geq 2$ be given and let $\mathcal{P} \subset \mathbb{N}$ denote the set of all prime numbers, then we define

$$\bar{p} := \bar{p}(K) := \inf \left\{ q \in \mathcal{P} \left| \sum_{p=2}^{q} \frac{p-1}{p} \in [K-1, K) \right. \right\}.$$ \hspace{1cm} (8)

Obviously, $\bar{p}$ is well-defined due to the divergence of the series. Since all denominators of the considered summands

$$\left\{ \frac{1}{p} \left| p \in \mathcal{P}, p \leq \bar{p} \right. \right\}$$

are relatively prime, the common denominator

$$h := h(K) := \prod_{p=2, \ p \in \mathcal{P}} p$$ \hspace{1cm} (9)

is given by the product of all these primes. Hence, the construction of $\bar{p}$ immediately implies that there exists $s := s(K) \in \mathbb{Z}_+$ with $s < h$ and

$$\sum_{p=2, \ p \in \mathcal{P}} \frac{p-1}{p} = K - 1 + \frac{s}{h}. \hspace{1cm} (10)$$

**Lemma 11.** Let $h$ and $s$ be given as in (9) and (10). Then, the following statements are true:

1. $s$ and $h$ are coprime.
2. $s > 0$.
3. $h - s > 1$.

**Proof.** We want to prove each assertion by contradiction:

1. Obviously,

$$\sum_{p=2, \ p \in \mathcal{P}} \frac{p-1}{p} = \sum_{p=2, \ p \in \mathcal{P}} \left( \frac{p-1}{h} \cdot r_p \right)$$

holds with

$$r_p := \frac{h}{p} = \prod_{q=2, \ q \in \mathcal{P}, q \neq p} q$$ \hspace{1cm} (11)

for all $p \in \Lambda := \Lambda(K) := \{ p \in \mathcal{P} \mid p \leq \bar{p} \}$.

Assuming that there is a common prime factor $c \in \mathcal{P}$ of $s$ and $h$, this number has
to satisfy $c \in \Lambda$. Then, actually, $c$ is a divisor of all $r_p$ with $p \in \Lambda$ and $p \neq c$. Consequently, $c$ is also a divisor of

$$(K - 1)h + s - \sum_{p=2,\, p \notin P \neq c}^p (p - 1)r_p$$

since all summands possess the prime factor $c$. Due to (10), we obtain

$$(K - 1)h + s - \sum_{p=2,\, p \notin P \neq c}^p (p - 1)r_p = (c - 1)r_c$$

stating that $c$ also divides $(c - 1)r_c$. Since $c \in \mathbb{P}$ holds we have $c \geq 2$, and $c$ cannot divide $c - 1$. Hence, $c$ divides $r_c = h/c$ implying that $c$ is, at least, a double prime factor of $h$ which contradicts to (9).

ii) Assuming that $s = 0$ holds, then

$$\frac{1}{h} \sum_{p=2,\, p \notin P}^p (p - 1)r_p = \sum_{p=2,\, p \notin P}^p \frac{p - 1}{p} = K - 1 \in \mathbb{N}$$

follows directly from (10) and (11). Hence, the numerator on the left hand side has to be a multiple of $h$. In particular, this numerator has to be divisible by each prime factor of $h$.

Now, we continue similar to the first part of this proof. Let $t$ be a prime factor of $h$, then $t \in \Lambda$ holds. All of the numerator’s summands of type $(p - 1)r_p$ with $p \in \Lambda$ and $p \neq t$ are divisible by $t$, but the summand $(t - 1)r_t$ is not. Thus, the numerator is no multiple of $t$ which gives the contradiction.

iii) Due to $s < h$, only the case $h - s = 1$ has to be excluded. Let $h - s = 1$, then (10) results to

$$\sum_{p=2,\, p \notin P}^p \frac{p - 1}{p} = K - 1 + \frac{h - 1}{h} = K - \frac{1}{h}.$$  

Now, we subtract the term $(\bar{p} - 1)/\bar{p}$ on the left hand side obtaining the sum

$$\sum_{p=2,\, p \notin P}^{\bar{p} - 1} \frac{p - 1}{p} = \sum_{p=2,\, p \notin P}^\bar{p} \frac{p - 1}{p} - \frac{\bar{p} - 1}{\bar{p}} = K - \frac{1}{h} - \frac{\bar{p} - 1}{\bar{p}} \geq K - \frac{1}{h} - \frac{\bar{p} - 1}{h} = K - 1$$

where $\bar{p} \leq h$ has been used, according to (9). But then, $\bar{p}$ is not minimal which gives the contradiction.
In what follows, we would like to construct an instance $E = (m, l, 1, b)$ of the divisible case in standard form satisfying $\lfloor l^\top b \rfloor = K$. Due to (10), our current item supply is not yet sufficient. Thus, we have to add some items in a smart way.

**Proposition 12.** There are positive integers $v, w \in \mathbb{N}$ satisfying

$$K - 1 + \frac{s}{h} + \frac{v}{w} = K + \frac{1}{hw}.$$

**Proof.** The assertion (12) can be transformed equivalently into

$$K - 1 + \frac{s}{h} + \frac{v}{w} = K + \frac{1}{hw} \iff sw + hv - hw = 1 \iff hv = (h - s)w + 1. \tag{13}$$

Now, we aim at constructing a number $g \in \mathbb{N}$ with

$$g \equiv 0 \mod h,$$

$$g \equiv 1 \mod (h - s),$$

$$g \geq 2$$

since this number allows the definition

$$v := \frac{g}{h}, \quad w := \frac{g - 1}{h - s}. \tag{15}$$

Then, (14) would lead to

$$hv = h \cdot \frac{g}{h} = g,$$

$$(h - s)w + 1 = (h - s) \cdot \frac{g - 1}{h - s} + 1 = g$$

which proves (12) due to the equivalence (13).

In the following, we show how the number $g$ can be constructed. As a consequence of Lemma 11 (i), the numbers $h$ and $h - s$ are coprime. By means of the extended Euclidean algorithm, we can find $\rho_1, \rho_2 \in \mathbb{Z}$ such that

$$\rho_1 h + \rho_2 (h - s) = \gcd(h, h - s) = 1$$

holds. Setting $\gamma := \rho_1 h$, we obtain

$$\gamma \equiv \rho_1 h \equiv 0 \mod h$$

by construction. Due to $\gamma = \rho_1 h = 1 - \rho_2 (h - s)$, also

$$\gamma \equiv 1 - \rho_2 (h - s) \equiv 1 \mod (h - s)$$

follows since $h - s > 1$ is true according to Lemma 11.

If $\gamma \geq 2$ holds, set $g := \gamma$, otherwise consider

$$g := \gamma + \delta \lcm(h, h - s)$$
with
\[ \delta := \inf \{ \xi \in \mathbb{N} \mid \gamma + \xi \lcm(h, h-s) \geq 2 \} = \min \{ \xi \in \mathbb{N} \mid \gamma + \xi \lcm(h, h-s) \geq 2 \}. \]

Since \( h \) and \( h-s \) are coprime, \( \lcm(h, h-s) = h(h-s) \) holds stating that \( g \) satisfies the conditions \([14]\) because \( g \geq 2 \) and
\[
\begin{align*}
g & \equiv \gamma + \delta \lcm(h, h-s) \equiv \gamma + \delta h(h-s) \equiv 0 \mod h, \\
g & \equiv \gamma + \delta \lcm(h, h-s) \equiv \gamma + \delta h(h-s) \equiv 1 \mod (h-s)
\end{align*}
\]
hold. Defining \( v \) and \( w \) as in \([15]\), the claim has been proved.

In the following lemma, we state some properties of \( v \) and \( w \) that will be needed for the construction of non-IRDP instances.

**Lemma 13.** The following statements are true:

i) Let \( C \in \mathbb{N} \) be a given constant. Then, \( w \) in \([12]\) can be chosen such that \( w > C \) holds.

ii) The numbers \( w \) in \([12]\) and \( h \) in \([9]\) are coprime.

iii) The numbers \( v \) and \( w \) in \([12]\) can be chosen such that \( v < w \) holds.

iv) There exists a sequence \( \{(v(K), w(K))\}_{K \geq 2} \) in \( \mathbb{N}^2 \) with:

(a) The element \((v(K), w(K))\) satisfies \([12]\) for all \( K \geq 2 \).

(b) The sequence \( \{w(K)\}_{K \geq 2} \) is monotonically increasing with \( w(2) > 1 \).

(c) \( v(K) < w(K) \) holds for all \( K \geq 2 \).

**Proof.**

i) Let \( C \in \mathbb{N} \) with \( w \leq C \) be given. Then, we set
\[
\begin{align*}
g' & := g + \lcm(h, h-s) = g + h(h-s) \\
\Rightarrow v' & = \frac{g'}{h} = v + (h-s), \\
w' & = \frac{g'-1}{h-s} = w + h
\end{align*}
\]
for the corresponding \( g \in \mathbb{N} \) from the proof of Proposition \([12]\). Hence, \( w' \geq w + 2 \) holds and the condition \([12]\) is still satisfied due to
\[
K - 1 + \frac{s}{h} + \frac{v'}{w'} = K + \frac{1}{hw'}
\]

\[\iff hv' = (h-s)w' + 1\]
\[\iff h(v + h - s) = (h-s)(w + h) + 1\]
\[\iff hv = (h-s)w + 1\]
\[\iff K - 1 + \frac{s}{h} + \frac{v}{w} = K + \frac{1}{hw}\]

where the last line is true by construction of \( v \) and \( w \). Since \( w \) increases by at least 2 in every step, \( w > C \) is obtained after a finite number of steps.
ii) Assuming that $w$ and $h$ possess a common prime factor $p \in \mathbb{P}$, then
\[ \gcd(h, w) \geq p \geq 2 \]
holds. Due to (15), this implies $\gcd(g, g - 1) \geq p \geq 2$ which gives the contradiction.

iii) Note that the method of part (i) of this proof cannot be applied directly since $v$ increases in each step, too. Assuming that $v \geq w$, we have to consider two cases:

(a) **Case 1:** $v = w$

In this case, we obtain
\[ hv = (h - s)w + 1 \quad \text{(v = w)} \iff hv = (h - s)v + 1 \iff sv = 1 \iff s = v = 1 \]
by means of (12) and (13) implying that $v = 1$ holds due to (12). In analogy to the construction in part (i), we set $v' = v + (h - s)$ and $w' = w + h$. Due to $h - s < h$, we have $v' < w'$, and the condition (12) is still satisfied.

(b) **Case 2:** $v > w$

In this case, we obtain
\[ hv = (h - s)w + 1 = hw - sw + 1 \quad \text{(< v = w)} \iff hv - sw + 1 \iff sw < 1 \]
by means of (12) and (13). Thus, at least one of the variables $s$ and $w$ has to equal zero. By definition and Lemma 11(ii), respectively, $w = 0$ and $s = 0$ are both impossible stating that this second case cannot occur.

iv) This claim follows directly from the previous parts of this proof. Due to Proposition 12, for each $K \geq 2$ there are natural numbers $v(K)$ and $w(K)$ satisfying (12). Furthermore, according to Lemma 13(iii), these numbers can be chosen such that $v(K) < w(K)$ holds for all $K \geq 2$. Considering the resulting sequence $\{(v(K), w(K))\}_{K \geq 2}$, the properties (iv)(a) and (iv)(c) are satisfied by construction. Due to Lemma 13(i), for $K = 2$ we can choose $w(2)$ such that $w(2) > 1$ is true. By the same reason, $w(K) \geq w(K - 1)$ can be attained for each $K \geq 3$. Note that the necessary transformations (see part (i)) do not change the relation $v(K) < w(K)$ since $h - s < h$ holds. Thereafter, the sequence possesses all properties (a)-(c) of assertion (iv).

**Definition 1.** A sequence possessing the properties of Lemma 13(iv) is called $M$-sequence.

With the help of these preliminary studies, we can formulate and prove the main result of this section.

**Theorem 14.** Let $K \geq 2$ and an $M$-sequence $\{(v(K), w(K))\}_{K \geq 2}$ be given. Setting
\[
m = |\Lambda(K)| + 1, \\
l = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ \vdots & \ddots & \ddots \\ \bar{p}(K) & \bar{p}(K) & w(K) \end{pmatrix}, \\
L = 1, \\
b = (1, 2, 4, \ldots, \bar{p}(K) - 1, v(K)),
\]
we obtain an instance $E = (m, l, L, b)$ possessing the following properties:
i) $E$ is an instance of the divisible case in standard form with $\lfloor l^\top b \rfloor = K$.

ii) There is no pattern $a \in \mathbb{Z}_+^m$ with $a \leq b$ and

$$1 \leq l^\top a < 1 + \frac{1}{h(K)w(K)}.$$  \hfill (16)

iii) The gap can be calculated by

$$\Delta(E) = 1 + \frac{1}{h(K)w(K)},$$  \hfill (17)

e.i., $E$ is a non-IRDP instance.

Proof.

i) Due to $L = 1$, $E$ represents an instance of the divisible case in standard form. Besides, we have

$$l^\top b = \sum_{p=2}^{\bar{p}(K)} p \cdot \frac{1}{p} + \frac{v(K)}{w(K)} - 1 + \frac{s(K)}{h(K)} + \frac{v(K)}{w(K)} - 1 \quad \text{for some} \quad \xi \in \mathbb{Z}_+.$$  \hfill (10)

implying that $\lfloor l^\top b \rfloor = K$.

ii) Let $a \in \mathbb{Z}_+^m$ be a pattern satisfying (16). Obviously, the common denominator of all components of the vector $l$ is given by $h(K)w(K)$. Hence, a pattern can only have a total length of

$$1 + \frac{\xi}{h(K)w(K)}$$

for some $\xi \in \mathbb{Z}_+$. According to the presumption (16), only the case $\xi = 0$ is possible. Thus, it suffices to show that an exact pattern $a \in P_{E}^*$ with $a \leq b$ does not exist. Assuming that $a \in \mathbb{Z}_+^m$ is an exact pattern with $a \leq b$, i.e., $l^\top a = 1$ holds. Let

$$I(a) = \{ i \in \{ 1, \ldots, m \} \mid a_i > 0 \}$$

denote the set of active indices. Then,

$$l^\top a = 1 \iff \sum_{i=1}^{m} a_i l_i = 1 \iff \frac{1}{h(a)} \sum_{i \in I(a)} a_i r_i(a) = 1$$

holds where

$$h(a) := \prod_{j \in I(a)} \frac{1}{l_j}$$

denotes the common denominator of all lengths appearing in $a \in \mathbb{Z}_+^m$ and $r_i(a)$ is given by

$$r_i(a) := \prod_{j \in I(a), i \neq j} \frac{1}{l_j} = l_i h(a) \in \mathbb{N}$$
for all $i \in I(a)$, similar to (11).

In order to obtain $l^\top a = 1$, the numerator

$$\sum_{i \in I(a)} a_i r_i(a)$$

has to be a multiple of every denominator in the set

$$N(a) := \left\{ \frac{1}{l_i} \mid i \in I(a) \right\}.$$ 

Let $t \in N(a)$ be given, then there is $j \in I(a)$ with $t = l_j^{-1}$. Consequently, $t$ divides all summands of the numerator with $i \neq j$ since, in these cases, $t$ divides $r_i(a)$. However, $t$ is not a divisor of $a_j r_j(a)$ since, on the one hand, $a_j \leq b_j < l_j^{-1} = t$ holds, and, on the other hand, $r_j(a)$ and $l_j^{-1}$ are coprime. Thus, the numerator is not a multiple of $t$ implying that there does not exist any exact pattern $a \in \mathbb{Z}_m^+$ with $a \leq b$.

iii) Due to Lemma 4 and part (i) of this theorem, the considered instance $E$ has the optimal objective value

$$z^*_c(E) = K + \frac{1}{h(K) w(K)}.$$ 

Hence, it suffices to show that there is no pattern $a \in \mathbb{Z}_m^+$ with $a \leq b$ and

$$1 \leq l^\top a \leq 1 + \frac{1}{h(K) w(K)}.$$ 

According to part (ii) of this theorem, it only remains to check whether

$$l^\top a = 1 + \frac{1}{h(K) w(K)} \quad (18)$$

is possible or not. Therefore, we consider two cases depending on the value of $K$.

(a) **Case 1: $K = 2$**

In this case, $\bar{p}(2) = 3$ holds due to the equation

$$\frac{1}{2} + \frac{2}{3} = (K - 1) + \frac{1}{6}.$$ 

Furthermore, we have $m(2) = 3$, $h(2) = 6$, $s(2) = 1$ and $h(2) - s(2) = 5$.

According to our construction principle from Proposition 12, we aim at finding a natural number $g \geq 2$ satisfying

$$g \equiv 0 \mod 6, \quad g \equiv 1 \mod 5.$$ 

Obviously, the solutions of this system are given by $g = 6 + 30\xi$ with $\xi \in \mathbb{Z}_+$. Setting $\xi = 0$, i.e., $g = 6$, we obtain

$$w(2) = \frac{g - 1}{h(2) - s(2)} = \frac{5}{5} = 1.$$
contradicting property (b) of an M-sequence. Consequently, we have $g = 6 + 30\xi$ with $\xi \in \mathbb{N}$ leading to

$$w(2) = \frac{g - 1}{h(2) - s(2)} = \frac{5 + 30\xi}{5} = 1 + 6\xi \geq 7,$$

$$\implies \frac{1}{w(2)} \leq \frac{1}{7} \text{ and } \frac{1}{h(2)w(2)} = \frac{1}{6 \cdot (1 + 6\xi)} \leq \frac{1}{42}.$$

In order to attain a total length as in (18), every single item length $l_i$ ($i = 1, 2, 3$) has to be taken at least once since, otherwise, the common denominator would differ from $h(2)w(2)$. Observing that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{w(2)} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42} < 1,$$

we conclude that one item length has to appear at least twice in the pattern $a$. Obviously, this is not possible for $l_2 = \frac{1}{3}$ since then the permitted total length would be exceeded due to

$$\frac{1}{2} + \frac{2}{3} + \frac{1}{w(2)} > \frac{1}{2} + \frac{2}{3} = 1 + \frac{1}{6} > 1 + \frac{1}{42} = 1 + \frac{1}{h(2)w(2)}.$$

Hence, only $l_3 = w(2)^{-1}$ can appear more than once. Let $\alpha \in \mathbb{N}$ with $2 \leq \alpha \leq v(2)$ denote the number of items with length $l_3$ appearing in $a$. Then, we obtain

$$1 + \frac{1}{6w(2)} = \frac{1}{2} + \frac{1}{3} + \frac{\alpha}{w(2)} \iff \frac{\alpha}{w(2)} = \frac{6w(2) + 1 - 3w(2) - 2w(2)}{6w(2)} \iff 6\alpha = w(2) + 1$$

and, due to $w(2) = 1 + 6\xi$ for some $\xi \in \mathbb{N}$, this is equivalent to

$$6\alpha = 2 + 6\xi \iff \alpha - \xi = \frac{1}{3}.$$

Since $\alpha - \xi \in \mathbb{Z}$ has to hold there is no pattern $a$ with property (18) in the case $K = 2$.

(b) Case 2: $K \geq 3$

In this case, we have

$$\sum_{p=2,\ p \in \mathbb{P}} \frac{p - 1}{p} \geq 2,$$

i.e., the sum on the left hand side contains at least the fractions

$$\frac{1}{2} + \frac{2}{3} + \frac{4}{5} + \frac{6}{7}.$$
Due to (9), the inequality $h(K) \geq h(K - 1)$ holds. Furthermore, we have $w(K) \geq w(K - 1)$ according to property (b) of an M-sequence implying that $h(K)w(K) \geq h(K - 1)w(K - 1)$. In particular, this leads to
\[
\begin{align*}
  (K \geq 3) & \quad h(K)w(K) \\
  (w(3) > 1) & \quad \geq h(3)w(3) > 2 \cdot 3 \cdot 5 \cdot 7 = 210.
\end{align*}
\]  

We assume that there is a pattern $a \in \mathbb{Z}_+^m(K) = \{a \leq b\}$ satisfying (18). In order to attain this total length, every single item length $l_i (i = 1, \ldots, m(K))$ has to be taken at least once since, otherwise, the common denominator would differ from $h(K)w(K)$. But then, due to (19), the total length of these items satisfies
\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{w(K)} > 1 + \frac{37}{210} > 1 + \frac{1}{210} \geq 1 + \frac{1}{h(K)w(K)},
\]
contradicting (18). Hence, there is no such pattern.

Consequently, the optimal objective value of the ILP cannot equal $K$, i.e., $z^*(E) \leq K - 1$ holds. Due to Theorem 9 we also have $\Delta(E) < 2$ entailing $z^*(E) \geq K - 1$. Hence, the gap $\Delta(E)$ results to
\[
\Delta(E) = z^*_c(E) - z^*(E) = \left( K + \frac{1}{h(K)w(K)} \right) - (K - 1) = 1 + \frac{1}{h(K)w(K)}
\]
proving (17). In particular, $E$ is a non-IRDP instance.

We would like to point out that this construction can be used in a more general way to obtain a multitude of further non-IRDP instances.

**Remark 15.** Let $K \in \mathbb{N}$ be a natural number with $K \geq 2$, $\tilde{P} \subseteq \mathbb{P}$ a subset of the prime numbers and $f : \tilde{P} \to \mathbb{N}$ a mapping with $f(p) \leq p - 1$ for all $p \in \tilde{P}$ such that the series
\[
\sum_{p \in \tilde{P}} \frac{f(p)}{p}
\]
diverges. Defining, in analogy to (8),
\[
\bar{p}^{f,\tilde{P}} := \bar{p}^{f,\tilde{P}}(K) := \inf \left\{ q \in \tilde{P} \left| \sum_{p^2 \leq q, p \in \tilde{P}} \frac{f(p)}{p} \in [K - 1, K) \right. \right\}
\]
the whole theory presented in this section can be applied in a straightforward way to obtain non-IRDP instances.

Note that we used $\tilde{P} = \mathbb{P}$ as well as $f(p) = p - 1$ for all $p \in \tilde{P}$. Hence, we took the greatest possible denominators and also their maximum availability. Thus, for some fixed $K \geq 2$, the instances from Theorem 14 contain the minimum number $m$ of item types among all instances that can be generated by this method.
5 Some investigations on the maximum gap for the divisible case

In this section, based on the instances of Theorem 14, we would like to investigate the gap of the divisible case in more detail. In the context of one-dimensional cutting, the greatest known gap is given by $137/132$, see [Sch08, p. 69]. As a main result, we show that this gap can be surpassed in the skiving scenario by modifying the considered instances slightly.

As an initial point, remember that for an M-sequence \( \{(v(K), w(K))\}_{K \geq 2} \) the sequence
\[
\{h(K) \cdot w(K)\}_{K \geq 2}
\]
is monotonically increasing. Thus, the greatest gap is provided in the case \( K = 2 \) by the instance \( E^\star = (3, \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{7}\right), 1, (1, 2, 6)) \) which is known from Remark 3. Obviously, this gap results to
\[
\Delta(E^\star) = 1 + \frac{1}{42}.
\]

By adding new item lengths in a smart way, it is possible to increase the gap.

**Theorem 16.** Let \( E = (m, l, 1, b) \) be an instance of the divisible case in standard form with \( l^\top b = K + r \) for some \( K \in \mathbb{N} \) with \( K \geq 2 \) and some \( r \in \mathbb{R} \) with \( 0 \leq r < 1 \). Let \( \varepsilon_1, \varepsilon_2 > 0 \) exist satisfying
\[
\bar{\bar{z}}(E) := \min \{ l^\top a \mid a \in \mathbb{Z}^m_+, l^\top a \geq 1, a \leq b \} = 1 + r + \varepsilon_1, \tag{20}
\]
\[
\bar{z}(E) := \max \{ l^\top a \mid a \in \mathbb{Z}^m_+, l^\top a \leq 1, a \leq b \} = 1 - \varepsilon_2 \tag{21}
\]
then define
\[
\varepsilon := \min \{ \varepsilon_1, \varepsilon_2 \}.
\]

Now, let \( d \in \mathbb{N} \),
\[
\lambda = \left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_d}\right) \quad \text{and} \quad \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d
\]
be given with different \( \lambda_1, \ldots, \lambda_d \in \mathbb{N} \) satisfying
\[
\forall j \in \{1, \ldots, d\} : \quad \frac{1}{\lambda_j} \notin \{l_1, \ldots, l_m\} \tag{22}
\]
and
\[
\sum_{i=1}^d \frac{\beta_i}{\lambda_i} < \varepsilon. \tag{23}
\]
Then, the following assertions are true:

i) \( \Delta(E) = 1 + r \).
ii) Let $\tilde{E}$ denote the instance being formed by adding the lengths $\lambda$ and availabilities $\beta$ to $E$. Then, we have

$$\Delta(\tilde{E}) = 1 + r + \sum_{i=1}^{d} \frac{\beta_i}{\lambda_i}. \tag{24}$$

In particular, $\Delta(\tilde{E}) > \Delta(E)$ holds.

Proof.

i) Due to (20), it is not possible to find $K$ (disjoint) patterns among the given item supply, i.e., $z^*(E) \leq K - 1$ has to hold. According to Theorem 9, we also have $z^*(E) \geq K - 1$ leading to $\Delta(E) = 1 + r$.

ii) Let $d \in \mathbb{N}$,

$$\lambda = \left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_d}\right) \quad \text{and} \quad \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$$

with different $\lambda_1, \ldots, \lambda_d \in \mathbb{N}$ be given satisfying

$$\forall j \in \{1, \ldots, d\} : \frac{1}{\lambda_j} \notin \{l_1, \ldots, l_m\}$$

and

$$\sum_{i=1}^{d} \frac{\beta_i}{\lambda_i} < \varepsilon.$$ 

Obviously, it suffices to show that $\tilde{E}$ contains no pattern $(a, \alpha)^\top \in \mathbb{Z}_{m+d}^+$ with $(a, \alpha)^\top \leq (b, \beta)^\top$ and

$$1 \leq \begin{pmatrix} l \lambda^\top \\ a \alpha^\top \end{pmatrix} \leq 1 + r + \sum_{i=1}^{d} \frac{\beta_i}{\lambda_i}. \tag{25}$$

We assume that there exists such a pattern $(a, \alpha)^\top \in \mathbb{Z}_{m+d}^+$. Without loss of generality, $(a, \alpha)^\top$ can be considered as a minimal pattern. Let

$$I(\alpha) = \left\{ 1 \leq i \leq d \mid \alpha_i > 0 \right\}$$

denote the set of the new objects’ active indices. Then, $I(\alpha) \neq \emptyset$ holds since, otherwise, $a \in \mathbb{Z}_{m+}^+$ would be a pattern of $E$ with $a \leq b$ satisfying the inequality $1 \leq l^\top a < 1 + r + \varepsilon$, due to (25) and (23). But this is not possible since $\bar{\varepsilon}(E) = 1 + r + \varepsilon_1 \geq 1 + r + \varepsilon$ has to hold.

The minimality of $(a, \alpha)^\top$ leads to

$$1 \leq \begin{pmatrix} l \lambda^\top \\ a \alpha^\top \end{pmatrix} < 1 + \min \left\{ \frac{1}{\lambda_i} \mid i \in I(\alpha) \right\}.$$
Removing all item lengths belonging to \( \lambda \) from the pattern \((a, \alpha)^\top\) (setting \( \alpha_i = 0 \) for \( i \in I(\alpha) \)), we obtain a cutting pattern of \( E \) because of

\[
I^\top a < 1 + \min \left\{ \frac{1}{\lambda_i} \bigg| i \in I(\alpha) \right\} - \sum_{i \in I(\alpha)} \frac{\alpha_i}{\lambda_i} \leq 0
\]

This cutting pattern’s total length satisfies

\[
I^\top a = \left( \begin{array}{c} l \\ \lambda \end{array} \right)^\top \left( \begin{array}{c} a \\ \alpha \end{array} \right) - \sum_{i \in I(\alpha)} \frac{\alpha_i}{\lambda_i} \geq 1 - \sum_{i=1}^{\infty} \frac{\beta_i}{\lambda_i} > 1 - \varepsilon_2
\]

in contradiction to the presumption \( \bar{z}(E) = 1 - \varepsilon_2 \). Hence, there is no packing pattern \((a, \alpha)^\top \in \mathbb{Z}_+^{m+d}\) with \((a, \alpha)^\top \leq (b, \beta)^\top\) and the property \((25)\).

\[\Box\]

The following lemma states an important observation for our intended construction of instances with large gaps.

**Lemma 17.** Let \( E = (m, l, 1, b) \) be an instance of the divisible case in standard form satisfying \( l^\top b = 2 + r \) for some \( r \in \mathbb{R} \) with \( 0 \leq r < 1 \). Furthermore, let \( \varepsilon > 0 \) exist such that

\[
\bar{z}(E) := \min \left\{ I^\top a \big| a \in \mathbb{Z}_+^m, I^\top a \geq 1, a \leq b \right\} = 1 + r + \varepsilon
\]

holds. Then,

\[
\bar{z}(E) := \max \left\{ I^\top a \big| a \in \mathbb{Z}_+^m, I^\top a \leq 1, a \leq b \right\} = 1 - \varepsilon
\]

is also true.

**Proof.** According to the presumption \((26)\), the shortest packing pattern \( a^* \in \mathbb{Z}_+^m \) with \( a \leq b \) of \( E \) satisfies \( I^\top a^* = 1 + r + \varepsilon \). Choosing those objects that do not belong to \( a^* \), we obtain a feasible solution for the maximization problem \((27)\) with objective value

\[
2 + r - (1 + r + \varepsilon) = 1 - \varepsilon
\]

which leads to

\[
\bar{z}(E) \geq 1 - \varepsilon.
\]

Assuming that \( \bar{z}(E) > 1 - \varepsilon \) holds, then there would exist a cutting pattern \( \alpha^* \in \mathbb{Z}_+^m \) with \( I^\top \alpha^* > 1 - \varepsilon \). Considering those objects that do not belong to \( \alpha^* \), we obtain a packing pattern \( \tilde{a} \in \mathbb{Z}_+^m \) satisfying \( a \leq b \) and

\[
I^\top \tilde{a} = 2 + r - I^\top \alpha^* < 2 + r - (1 - \varepsilon) = 1 + r + \varepsilon
\]

in contradiction to the optimality of \( a^* \). Thus, \( \bar{z}(E) = 1 - \varepsilon \) has to hold.

\[\Box\]

As a direct consequence of this lemma, we have \( \varepsilon_1 = \varepsilon_2 = \varepsilon \) in Theorem \((16)\) for the case \( K = 2 \). This result plays an important role in the proof of the following construction method.
Corollary 18. Let $\delta \in \mathbb{R}$ with $0 \leq \delta < 1/42$ be given. Then, there exists an instance $E$ of the divisible case in standard form satisfying

$$1 + \frac{1}{21} > \Delta(E) \geq 1 + \frac{1}{42} + \delta.$$  \hspace{1cm} (28)

Proof. Consider the instance $E^*$ from the beginning of this section. There, we have $K = 2$, $r = 1/42$ and

$$\tilde{z}(E^*) = \frac{44}{42} = 1 + \frac{1}{21}$$

stating that $\varepsilon = 1/42$ holds. Due to the density of the rational numbers in $\mathbb{R}$, there are different lengths $\lambda_1, \ldots, \lambda_d \in \mathbb{N}$ and availabilities $\beta_1, \ldots, \beta_d \in \mathbb{N}$ with

$$\delta \leq \sum_{i=1}^{d} \frac{\beta_i}{\lambda_i} < \frac{1}{42}.$$  

Let $\tilde{E}^*$ denote the instance that is created by adding these objects to $E^*$. According to Theorem 16, its gap results to

$$\Delta(\tilde{E}^*) = 1 + \frac{1}{42} + \sum_{i=1}^{d} \frac{\beta_i}{\lambda_i} \geq 1 + \frac{1}{42} + \delta.$$  

Besides, we also have

$$\Delta(\tilde{E}^*) = 1 + \frac{1}{42} + \sum_{i=1}^{d} \frac{\beta_i}{\lambda_i} < 1 + \frac{1}{42} + \frac{1}{42} = 1 + \frac{1}{21}$$

proving the assertion. \hfill \Box

Hence, we presented a construction principle for instances of the divisible case whose gaps can be arbitrarily close to $\Delta(E) = 22/21$. Due to

$$\frac{22}{21} - \frac{137}{132} = \frac{3}{308} > 0,$$

the maximum known gap of the cutting stock problem’s divisible case can be surpassed if $\delta \in \mathbb{R}$ is chosen in such a way that

$$\frac{1}{42} - \frac{3}{308} < \delta < \frac{1}{42}$$

holds.
6 Conclusion and outlook

In this paper, we gave an introduction to the gap of the skiving stock problem from a theoretical point of view. As a starting point, we generalized a result of [Zak03] in order to obtain a general upper bound for the gap. Thereby, we also stated a first sufficient condition for instances possessing the MIRDP.

Thereupon, we studied the divisible case intensively. For these instances, we have proved, as a main result, the MIRDP by means of the FFD heuristic for the dual bin packing problem. Moreover, we presented a contraction principle obtaining an infinite number of non-equivalent non-IRDP instances of the divisible case and showed how this method can be used, in a more general way, to create further classes of examples. Additionally, we investigated the gap of these instances also stating a technique to increase the gap by means of adding new item lengths. As a main contribution, this method was shown to lead to instances with gaps arbitrarily close to $22/21$ which surpasses the best known gap for the divisible case of the related one-dimensional cutting stock problem.

Except for the continuous relaxation studied in this paper, there are other possible relaxations that are of theoretical and practical interest. One of these is the proper relaxation where, as an additional restriction, only minimal patterns $a \in P_{E}^{*}$ with $a \leq b$, so-called proper patterns, are considered. This relaxation has also been studied in one-dimensional cutting scenarios [NST99], [KRS13], so its practicability in terms of the skiving stock problem will be investigated in the near future.

Since we mainly focused on the divisible case it will be part of our future research to study the gap of arbitrary instances of the skiving stock problem. Therefore, we aim at improving the upper bound $m - 1$ introduced in this paper in order to get a bit closer to the MIRDP-conjecture. Another main objective is given by the investigation of further special classes of instances also considering those ones that have turned out to possess the MIRUP in the context of one-dimensional cutting.

References


Integer Rounding and Modified Integer Rounding for the Skiving Stock Problem


