A Comparative Study of the Arcflow Model and the One-Cut Model for one-dimensional Cutting Stock Problems

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Abstract

We consider the one-dimensional cutting stock problem which consists of determining the minimum number of given large stock rolls that has to be cut to satisfy the demands of certain smaller item lengths. Besides the standard pattern-based approach of Gilmore and Gomory, containing an exponentially large number of variables, several pseudo-polynomial formulations were proposed in the last decades. In particular, much research has been dealt with arcflow models, their relationship to the standard model, and possible reduction methods, whereas the one-cut approach did not attract that much scientific interest yet. In this paper, we aim to compare both alternative formulations from a theoretical and numerical point of view. Therefore, we first present how reduction methods, partly originating from arcflow considerations, can successfully be transferred to the one-cut context. Furthermore, we derive relations between the numbers of variables and constraints in both models, and investigate their influences in numerical simulations. As a theoretical main contribution, we prove the equivalence of the continuous relaxations of the one-cut model and the pattern-based model.

Key words: Cutting and Packing, Cutting Stock Problem, Arcflow Model, One-Cut Model

1 Introduction and Preliminaries

The term cutting and packing addresses a wide variety of (mostly NP-hard) combinatorial optimization problems with high relevance in many fields of practice. Despite containing a large number of very diversified problem formulations, many of these problems share a common basic structure: select a subset of (small) items and assign it to larger objects by respecting some (geometrical) constraints. Note that, from a pure mathematical point of view, cutting and packing are often closely related since they principally describe the same process, but from a different perspective. Nevertheless, both terms are well-established in
literature to better refer to the particular application where the considered problem comes from. In most cases, the general objective is to minimize the waste, i.e., the portion of material that cannot be used for the intended purpose. Based on the typology of [24], (almost) all cutting and packing problems can be classified into:

- **Input Minimization Problems**, where as few resources as possible shall be used to achieve a predefined goal, and

- **Output Maximization Problems**, where a certain objective criterion has to be satisfied as good as possible by means of a given number of resources.

As the case may be, the economic component (i.e., lower production costs in a broad sense) and sustainability issues (i.e., less waste of raw material in general) of all these tasks are obvious.

In this paper we focus on the one-dimensional cutting stock problem (CSP): find the minimum number of large stock rolls of length \( L \), that has to be cut to satisfy the demands \( b_i \) of given smaller item lengths \( l_i \) \((i \in I := \{1, \ldots, m\})\). By introducing \( l = (l_1, \ldots, l_m)^\top \) and \( b = (b_1, \ldots, b_m)^\top \) we can refer to a specific cutting stock problem by means of the instance \( E = (m, l, L, b) \). Without loss of generality, we may assume:

(i) All input-data are positive integers.

(ii) The item lengths are of strictly decreasing order, i.e., \( L > l_1 > l_2 > \ldots > l_m > 0 \). If such an order is not given from the beginning, we can easily obtain it by a sorting algorithm, e.g. merge sort, in \( \mathcal{O}(m \cdot \log m) \) operations.

(iii) The equation \( L = \max \{l^\top a \mid a \in \mathbb{Z}_+^m, l^\top a \leq L, a \leq b\} \) is satisfied\(^1\). Otherwise \( L \) could be shortened.

(iv) The inequality \( l_1 + l_m \leq L \) holds. Otherwise, it is possible to discard the items of length \( l_1 \), if the optimal objective value of the remaining instance is increased by \( b_1 \).

The CSP is one of the most important representatives in combinatorial optimization (see [4, Fig. 1] for the development of related publications); the study of its structure and applications already started in 1939, when Kantorovich [9] formulated the first model to cope with that problem. Therein, based on an upper bound for the number of stock rolls, an assignment model with binary and integer variables is proposed. Unfortunately, this approach has turned out to possess two major drawbacks: a very weak continuous relaxation \(^2\), and a huge number of symmetries arising from permutations. Both points may be very critical when trying to tackle this model by means of branch-and-bound-techniques, see [21] or [23]; hence most authors forbear from doing further research regarding this approach.

In 1961, Gilmore and Gomory introduced a pattern-based approach [6], hereinafter referred to as the standard model, that overcomes these disadvantages. Thereby, the set of (cutting)

\(^1\)Note that \( a \leq b \) means that \( a_i \leq b_i \) holds for all \( i \in I \).

\(^2\)Nevertheless, the scientific work of L. Kantorovich in the field of optimally allocating given resources has been honored with the Nobel prize in economics (1975).
patterns for a given instance $E = (m, l, L, b)$ results to $P(E) = \{a \in \mathbb{Z}_m^+ \mid l^\top a \leq L\}$. More precisely, only maximal cutting patterns, described by

$$P^* := P^*(E) := \{a \in P(E) \mid l^\top a + l_m > L\},$$

have to be considered. Let $J^* := J^*(E)$ denote an appropriate index set of $P^*(E)$, and let $x_j \in \mathbb{Z}_+$ indicate how often pattern $a_j^i$ ($j \in J^*$) is applied within the considered (feasible) solution, then the standard model is given by

**Standard Model of the CSP**

$$z^S = \sum_{j \in J^*} x_j \rightarrow \min$$

s.t. 

$$\sum_{j \in J^*} a_{ij} x_j \geq b_i, \quad i \in I, \quad (1)$$

$$x_j \in \mathbb{Z}_+, \quad j \in J^*. \quad (2)$$

The objective function minimizes the total number of used stock rolls, whereas constraints (1) ensure that the item demands are met. In most cases, particularly for instances of practical meaningful size, it is not reasonable or even not possible to have all patterns available prior to the optimization process due to the huge cardinality of $P^*(E)$. Hence, common solvers, like CPLEX, usually cannot be applied to tackle this model. However, as already stated in [6], at least the continuous relaxation of the standard model can be solved efficiently by column generation. In order to solve the ILP, branch-and-price (see [1] or [23]) techniques can be applied. Note that, in this case, the computational behavior strongly depends on the choice of an appropriate branching rule.

A further way to tackle the integer problem consists in the consideration of other modeling approaches, most notably the arcflow model [22] and the one-cut model [7]. Observe that modeling frameworks on the basis of graph theory have been investigated and successfully applied in different fields of discrete optimization or operations research. One of the earliest references is given by [25] where theoretical foundations on the relationship between certain integer linear programs and corresponding flow-formulations have been proposed. Although their numerical behavior and benefits are not discussed in detail, some properties suggesting computational advantages of such formulations were presented therein. In the context of cutting and packing, arc-flow models are a well known tool for the exact solution of the cutting stock problem [22] and the neighboring skiving stock problem [13, 14]. In the last years, much research has been done to investigate and improve the corresponding models [3] and algorithms [2], even with respect to related problems.

Contrary to that, the one-cut model [7] attracted significantly less scientific interest so far. Indeed, the only important contribution known in literature is due to Stadtler [20] who investigated the structure of the original model in more detail, and presented an extension to Dyckhoff’s original approach. In that paper the author further stated:

“This leads to the conclusion, that all cutting stock problems that can be solved by one-cut models could also be solved by the column generation approach for the complete-cut model (but not vice versa).”

Moreover, Valério de Carvalho [22] pointed out:
"The number of variables in one-cut models is pseudopolynomial, and does not grow explosively as in the classical approach. That does not mean that the model is amenable to an exact solution by a good integer LP code, due to the symmetry of the solution space."

Both statements may have contributed to the fact that this model has almost been excluded from further scientific discussions regarding the cutting stock problem. Hence, we will take these two arguments as a starting point for our considerations. In a first step we propose several reduction methods that are useful to decrease the numbers of variables and constraints, as well as the symmetry of one-cut models. Furthermore, we provide a constructive proof for the equivalence of the (reduced) one-cut model and the classical pattern-based approach that also holds for the respective continuous relaxations. Thereby, we prove Stadtler’s argument to be (at least partly) wrong. To our best knowledge, this relationship (for the LP models) has not been established before in literature. Based on these two contributions, the one-cut model shall be put in its true light when lastly comparing its complexity to that of the well-known arcflow formulation.

This paper is organized as follows: the next section deals with a repetition of the arcflow model of [22], and proposes the most important reduction methods that can be applied in that context. Moreover, we introduce an alternative proof that states its equivalence to the formulation of Gilmore and Gomory. Afterwards, we show how these concepts can successfully be applied in the setting of one-cut models. In a final step, we compare both improved formulations by means of theoretical and numerical considerations.

## 2 The Arcflow Model

The general idea of the arcflow model is based on the observation that determining a valid solution for a single stock length $L$ can also be modeled as the problem of finding a path in an acyclic graph with $L + 1$ vertices. In the original version [22], this graph $G = (V, E)$ is given by the set of vertices $V = \{0, 1, \ldots, L\}$, and the set of arcs $E = E_1 \cup E_2$ consisting of

$$
E_1 = \{(p, q) \in V \times V | q - p \in \{l_1, \ldots, l_m\}\} \quad \text{and} \quad E_2 = \{(p, L) \in V \times V | p > L - l_m\}.
$$

Thereby, an arc $(p, q) \in E_1$ represents the positioning of one single item of length $q - p = l_i \in \{l_1, \ldots, l_m\}$ with its left point at vertex $p \in V$, whereas an arc $(p, L) \in E_2$ refers to some trim loss. In this framework, a solution for a single stock length $L$ corresponds to the flow of one unit from the source 0 to the sink $L$. Hence, a path carrying a larger flow can be interpreted as using the same cutting packing multiple times.

Let $x_{pq} \in \mathbb{Z}_+$ denote how often arc $(p, q) \in E$ is used within the considered (feasible)

\footnote{Note that the main focus of this paper is given by theoretical investigations on equivalence results and possible improvements, particularly with respect to one-cut formulations. Although numerical simulations may underline the corresponding observations, we intend to use them as an auxiliary tool rather than presenting a competitive and thorough computational study involving all available codes and approaches from literature.}
Arctflow Model

\[
\begin{align*}
    z^{AF} &= \sum_{(0,q) \in E} x_{0q} \to \min \\
    \text{s.t.} && \sum_{(p,q) \in E} x_{pq} = \sum_{(q,r) \in E} x_{qr}, & q \in \{1, \ldots, L - 1\}, \\
    && \sum_{(p,q) \in E, q - p = l_i} x_{pq} \geq b_i, & i \in I, \\
    && x_{pq} \in \mathbb{Z}_+, & (p, q) \in E.
\end{align*}
\]

Note that, as a first tiny reduction step, the redundant variable \(z\) (and two corresponding constraints of type (4)) appearing in the original model \([21]\) have been deleted. Constraints (4) can be interpreted as a flow conservation: at every interior vertex the number of incoming items has to equal the number of outgoing items. By the flow decomposition property, nonnegative flows can be represented by paths and cycles. Since \(G\) is acyclic, any integer solution can be decomposed into directed paths from the source node to the terminal node, i.e., into a set of feasible cutting patterns. Thus, the objective function minimizes not only the items placed at the node \(v_0 = 0\), but also the total number of used stock rolls of length \(L\). Conditions (5) ensure that the given demands are satisfied. In this arctflow model, the numbers of variables and constraints are \(O(mL)\) and \(O(m + L)\), respectively. Precise numbers are stated in the following lemma:

**Lemma 1.** The arctflow model contains \(m(L + 1) - \sum_{i=1}^{m-1} l_i - 1\) variables and \(L - 1 + m\) constraints.

**Proof.** Due to \(|I| = m\) and \(|\{1, \ldots, L - 1\}| = L - 1\), the second claim is obvious. In order to count the number of variables, we refer to the arcs in \(E_1\) and \(E_2\) separately. For this purpose, note that each item \(i \in I\) can begin exactly at the vertices \(\{0, \ldots, L - l_i\}\), i.e., there are \(L - l_i + 1\) possible positions leading to

\[
|E_1| = \sum_{i=1}^{m} (L - l_i + 1) = m(L + 1) - \sum_{i=1}^{m} l_i.
\]

As regards the set \(E_2\), observe that a loss arc can only start at the nodes \(\{L - l_m + 1, \ldots, L - 1\}\), resulting to \(|E_2| = l_m - 1\). Both results together imply the claimed equation for the number of variables. \(\square\)

Consequently, this model possesses significantly less variables, but far more constraints in exchange, compared to the pattern-based approach. Valério de Carvalho has verbally enumerated several criteria to reduce the model’s complexity \([21]\). Since we aim at transferring them later to the one-cut model, we will repeat the most important reduction principles and show how they can explicitly be incorporated in the model formulation itself.

i) **Potential allocation points:**

   Obviously, we only need to consider vertices that are integer linear combinations of
the given item lengths, since all other nodes cannot lie on a path \( s = (v_0, v_1, \ldots, v_k) \) from \( v_0 = 0 \) to \( v_k = L \). Thus, in general, we can replace our set of vertices by a reduced one, i.e., we define

\[
S(l, L) := \{ v \in \mathcal{V} \mid \exists a \in \mathbb{Z}_m^+ : l^\top a = v \}.
\]

Note that, to some extent, this property refers to the observation [21, Criterion 3].

ii) Removing loss arcs:

Instead of urging all paths to end at vertex \( L \), the consideration of more than one sink within the graph is also possible. This construction has been proved beneficial in the related one-dimensional skiving stock problem [13, 15], too. Hence, we can remove all loss arcs and allow a path to end at any vertex

\[
\mathcal{L} := \{ L_1 < \ldots < L_t \} := S(l, L) \cap \{ L - l_m + 1, \ldots, L \}.
\]

Note that, due to our preliminary assumptions (iii), we always have \( L_t = L \). Additionally, the flow conservation constraints only have to be satisfied for \( q \in \mathcal{I} := S(l, L) \setminus (\mathcal{L} \cup \{0\}) \). To our best knowledge, this reduction method has not been attempted for the arcflow model of the cutting stock problem yet.

iii) Breaking symmetry:

If we want to assign a path \( s \) in \( \mathcal{G} \) to a (maximal) cutting pattern \( a \in P^*(E) \), this identification is not unique since \( a \) does not contain any information about the particular arrangement of the items. We therefore like to restrict our investigations to monotonically decreasing paths, i.e., paths whose corresponding item lengths are sorted in descending order. Based on the idea of Scheithauer [19], we define an index \( \mu(p) \in I \cup \{0\} \) by

\[
\mu(p) := \begin{cases} 
0, & \text{if } p = 0, \\
\min \{ i \in I \mid p - l_i \in S(l, L), i \geq \mu(p - l_i) \}, & \text{if } p \in \mathcal{I},
\end{cases}
\]

for each \( p \in S(l, L) \setminus \mathcal{L} \). Note that the minimum in \((7)\) is well defined due to the definition of \( S(l, L) \). Moreover, this idea formalizes the verbal contribution of [21, Criteria 1,2].

Based on these reduction principles, we obtain a graph \( \mathcal{G}' = (\mathcal{V}', \mathcal{E}') \) of fewer complexity, where \( \mathcal{V}' := S(l, L) \) and

\[
\mathcal{E}' := \{ (p, q) \in \mathcal{V}' \times \mathcal{V}' \mid \exists i \in I : i \geq \mu(p), q - p = l_i \}
\]

hold.

\(^4\)Observe that the nearby idea of adding the condition \( a \leq b \) to the definition of \( S(l, L) \) would correspond to the consideration of the proper relaxation, see [10] or [16] for some related work.

\(^5\)Note that it is much easier to determine the lengths of all maximal cutting patterns than computing the lengths of all minimal packing pattern for the skiving stock problem. This is mainly due to the fact, that a maximal cutting pattern \( a \in \mathbb{Z}_m^+ \) can be characterized by the linear inequalities \( l^\top a - l_m < l^\top a \leq L \), whereas such a compact condition is not available in the skiving scenario.
Remark 2. The introduction of $\mu$ does not solve the problem of equivalent paths within the graph $G'$ entirely. Consider, by way of example, the instance $E = (2, (4, 2), 8, (2, 2))$ where, obviously, $\mu(0) = 0$, $\mu(2) = 2$ and $\mu(4) = 1$ hold. Subsequently, $G'$ especially contains the two paths $0 \to 2 \to 4 \to 8$ and $0 \to 4 \to 6 \to 8$ which both refer to the pattern $a = (1, 2)^\top$. Nonetheless, this method leads to a significant reduction of the set of arcs, see Section 5 for corresponding computational results.

For the sake of a better comprehensibility of the upcoming model, we introduce the sets

$$A^+(q) := \{p \in V' \mid (p, q) \in E'\},$$
$$A^-(q) := \{r \in V' \mid (q, r) \in E'\},$$

for every $q \in V'$ and the set

$$E(i) := \{(p, q) \in E' \mid q - p = l_i\}$$

for every $i \in I$. Note that the cases $A^+(q) = \emptyset$ and $A^-(q) = \emptyset$ are possible for some $q \in V'$. We are now able to state the

(Reduced) Arcflow Model

$$z^{AF} = \sum_{q \in A^-(0)} x_{0q} \to \min$$

s.t.  

$$\sum_{p \in A^+(q)} x_{pq} = \sum_{r \in A^-(q)} x_{qr}, \quad q \in I,$$  

$$\sum_{(p, q) \in E(i)} x_{pq} \geq b_i, \quad i \in I,$$  

$$x_{pq} \in \mathbb{Z}_+, \quad (p, q) \in E'.$$

Example 1. Consider the instance $E = (3, (5, 3, 2), 12, (2, 4, 6))$ with $L = \{11, 12\}$. Then, the (reduced) arcflow graph $G'$ consists of 12 vertices and 18 arcs, as depicted in Fig. 1.

![Figure 1: (Reduced) arcflow graph $G'$ for the instance $E = (3, (5, 3, 2), 12, (2, 4, 6))$.](image)

Valério de Carvalho has proved the equivalence between the (original) arcflow model [21, 22] and the pattern-based approach of Gilmore and Gomory by means of polyhedral theory. Remarkably, this result also holds for the continuous relaxations of both models. However, for the sake of completeness and as an alternative approach, we will provide a constructive proof showing that this equivalence is maintained after having improved the arcflow model. We emphasize that, due to Remark 2, an argumentation on the basis of equivalence classes of paths in $G'$ is inevitable.
Theorem 3. Let \( x \) be a feasible solution of the reduced arcflow model with objective value \( z(x) \). Then there is a feasible solution \( \tilde{x} \) of the standard model with the same objective value. In particular,

\[
(z^S)^* \leq (z^{AF})^*
\]

holds for the optimal values \((z^S)^*\) and \((z^{AF})^*\) of the standard and the reduced arcflow model, respectively.

Proof. Let \( x \in \mathbb{Z}_+^{|E'|} \) be a feasible solution of the reduced arcflow model with objective value \( z(x) \). Consider the set \( \Gamma \) of all paths in \( G' = (V', E') \) from \( v_0 \) to an element of \( L \). For each \( \gamma \in \Gamma \) we define a variable \( \lambda_\gamma \) stating how often the respective path is used in the feasible solution \( x \). Due to \( x \in \mathbb{Z}_+^{|E'|} \), each variable \( \lambda_\gamma \) (\( \gamma \in \Gamma \)) is a nonnegative integer.

Since constraints (8) hold for the feasible solution \( x \), the objective value \( z(x) \) can be obtained by

\[
\sum_{\gamma \in \Gamma} \lambda_\gamma = z(x). \tag{11}
\]

For each \((p, q) \in E'\) and \( \gamma \in \Gamma \) we introduce a decision variable by means of

\[
x^\gamma_{pq} := \begin{cases} 1, & \text{if the arc } (p, q) \text{ is contained in path } \gamma, \\ 0, & \text{otherwise}. \end{cases}
\]

Let the components of the vector

\[
a_\gamma := (a^\gamma_1, \ldots, a^\gamma_m) \top \in \mathbb{Z}_+^m
\]

denote the number of items of type \( i \in I \) that are contained in path \( \gamma \in \Gamma \). Since each path corresponds to a maximal pattern we obtain \( l^\top a_\gamma \in L \) and \( a_\gamma \in P^*(E) \). According to this construction,

\[
a^\gamma_i = \sum_{(p, q) \in E(i)} x^\gamma_{pq} \tag{12}
\]

holds for every \( i \in I \) and \( \gamma \in \Gamma \). We now define an equivalence relation \( \sim \) on the set \( \Gamma \) by means of

\[
\gamma_1 \sim \gamma_2 \iff a^\gamma_1 = a^\gamma_2,
\]

i.e., two paths are equivalent if and only if they contain the same number of each item length. Let

\[
\varphi : \Gamma / \sim \rightarrow \Gamma, [\gamma] \mapsto \varphi([\gamma]) = \gamma^*
\]

be a mapping assigning each equivalence class to its unique representative that is indicated by monotonically decreasing item lengths. Note that this mapping is not surjective in general, cf. Remark 2. We therefore consider \( \Gamma^* := \text{Im}(\varphi) \subseteq \Gamma \) and define the bijection

\[
\kappa : J^* \rightarrow \Gamma^*, j \mapsto \kappa(j),
\]

where \( \kappa(j) \) corresponds to the path \( \gamma^* \in \Gamma^* \) that uniquely belongs to the maximal pattern \( a^j \in P^*(E) \). In particular, we obtain

\[
a_{ij} = a^\kappa(j). \tag{13}
\]
Defining
\[ \tilde{\lambda}^\gamma := \sum_{s \in [\gamma]} \lambda^s \] (14)
for each \( \gamma^* \in \Gamma^* \), we claim that an appropriate feasible solution \( \tilde{x} \in \mathbb{Z}^{J^*} \) of the standard model is given by
\[ \tilde{x}_j := \tilde{\lambda}^{\kappa(j)} \] (15)
for all \( j \in J^* \). Indeed, we obtain:

i) Since \( \lambda^\gamma \in \mathbb{Z}_+ \) holds each \( \tilde{x}_j \) \( (j \in J^*) \) is a nonnegative integer.

ii) The objective value results from
\[
\sum_{j \in J^*} \tilde{x}_j \quad \text{(partition)} \quad \sum_{j \in J^*} \tilde{\lambda}^{\kappa(j)} \quad \text{(\( \kappa \) bijective)} \quad \sum_{\gamma^* \in \Gamma^*} \sum_{\gamma^* \in \Gamma^*} \sum_{s \in [\gamma^*]} \lambda^s 
\]
\[ = \sum_{\gamma \in \Gamma} \lambda^\gamma \] (11) \( z(x) \).

iii) Furthermore, \( \tilde{x} \) satisfies constraint (11) since we obtain
\[
\sum_{j \in J^*} a_{ij} \tilde{x}_j \quad \text{(\( \kappa \) bijective)} \quad \sum_{j \in J^*} a_{ij} \tilde{\lambda}^{\kappa(j)} \quad \sum_{\gamma^* \in \Gamma^*} \sum_{\gamma^* \in \Gamma^*} \sum_{s \in [\gamma^*]} a_{ij}^s \lambda^s 
\]
\[ = \sum_{\gamma \in \Gamma} \sum_{\gamma \in \Gamma} \sum_{(p,q) \in E(i)} x_{pq}^\gamma \lambda^\gamma \] (12) \[ = \sum_{\gamma \in \Gamma} \sum_{\gamma \in \Gamma} \sum_{(p,q) \in E(i)} x_{pq}^\gamma \lambda^\gamma \] (9) \[ \geq b_i \]
for each \( i \in I \). Note that in (\( \ast \)) the property
\[ \forall \gamma^* \in \Gamma^* \forall s \in [\gamma^*] : a_{ij}^s = a_{ij}^s \]
and in (\( \diamond \)) the property
\[ \forall (p,q) \in E' : x_{pq} = \sum_{\gamma} x_{pq}^\gamma \lambda^\gamma \]
were used, respectively.

Thus, the theorem is proved. \( \square \)

Obviously, this theorem still remains true if we consider the continuous relaxations of both models. Only the assertion \( \lambda^\gamma \in \mathbb{Z}_+ \) has to be replaced by \( \lambda^\gamma \geq 0 \) \( (\gamma \in \Gamma) \) which leads to \( \tilde{x}_j \geq 0 \) \( (j \in J^*) \).
Corollary 4. Let \( x \) be a feasible solution of the continuous relaxation of the reduced arcflow model with objective value \( z(x) \). Then there is a feasible solution \( \bar{x} \) of the continuous relaxation of the standard model with the same objective value. In particular,

\[
(z_{rel}^{S})^* \leq (z_{rel}^{AF})^*
\]

holds for the optimal values of both models.

We now focus on the reverse direction starting with a feasible solution of the standard model.

Theorem 5. Let \( x \) be a feasible solution of the standard model with objective value \( z(x) \). Then there is a feasible solution \( \tilde{x} \) of the reduced arcflow model with the same objective value. In particular,

\[
(z^{S})^* \geq (z^{AF})^*
\]

holds for the optimal objective values.

Proof. Let \( x = (x_1, \ldots, x_{|J^*|})^T \in \mathbb{Z}_{+}^{|J^*|} \) be a feasible solution of the standard model with objective value \( z(x) \). For each arc \((p,q) \in \mathcal{E}'\) and pattern \( j \in J^*\) we define a decision variable

\[
y_{pq}^j := \begin{cases} 
1, & \text{if path } \kappa(j) \text{ contains an item of length } q - p \text{ starting at vertex } p, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \kappa \) represents the bijection from the last proof. Let \( q \in \mathcal{I} \) and \( j \in J^* \) be given. Then, we obtain two fundamental properties:

(P1) If \( y_{pq}^j = 1 \) holds for some \((p,q) \in \mathcal{E}'\), then there is a unique index \( \tilde{s} \in A^- (q) \)

such that \( y_{q\tilde{s}}^j = 1 \) holds and the assertion \( y_{qs}^j = 0 \) is true for all \( s \in A^- (q) \) with \( s \neq \tilde{s} \).

(P2) If for all

\[
p \in A^+(q)
\]

the assertion \( y_{pq}^j = 0 \) holds, then we obtain \( y_{qs}^j = 0 \) for all \( s \in A^- (q) \).

For each pair \((p,q) \in \mathcal{E}'\) we define

\[
\bar{x}_{pq} := \sum_{j \in J^*} y_{pq}^j x_j
\]

claiming that this is a feasible solution of the arcflow model with appropriate objective value. Since \( x_j \in \mathbb{Z}_+ \ (j \in J^*) \) holds constraints (10) are satisfied. Furthermore, we can state:
i) Due to $A^-(0) = \{l_1, \ldots, l_m\}$ the objective value results to

\[
\sum_{q \in A^-(0)} \tilde{x}_{0q} = \sum_{i \in I} \sum_{j \in J^*} y_{0i}^j \tilde{x}_{ij} = \sum_{j \in J^*} x_j \sum_{i \in I} y_{0i}^j = \sum_{j \in J^*} x_j = z(x).
\]

Note that for each $j \in J^*$ the assertion \[\sum_{i \in I} y_{0i}^j = 1\] is true since every path $\kappa(j)$ has to start with a uniquely determined item length.

ii) We now verify constraints (9). Let $i \in I$ be arbitrarily chosen. Then we obtain

\[
\sum_{(p,q) \in E(i)} \tilde{x}_{pq} = \sum_{(p,q) \in E(i)} \sum_{j \in J^*} y_{pq}^j x_j = \sum_{j \in J^*} x_j \sum_{(p,q) \in E(i)} y_{pq}^j = \sum_{j \in J^*} \sum_{p \in A^+(q)} a_{ij} \geq b_i.
\]

iii) Let $q \in I$ be arbitrarily chosen, then we can use assertions $(P_1)$ and $(P_2)$. Observe that the inequality

\[
\sum_{p \in A^+(q)} y_{pq}^j \leq 1
\]

holds for all $j \in J^*$ since in each path $\kappa(j)$ there is at most one item length that can end at vertex $q$. Hence,

\[
\sum_{p \in A^+(q)} \tilde{x}_{pq} = \sum_{p \in A^+(q)} \sum_{j \in J^*} y_{pq}^j x_j = \sum_{j \in J^*} x_j \sum_{p \in A^+(q)} y_{pq}^j
\]

follows. Since (17) holds there are two possible cases for each $j \in J^*$ :

**Case 1:** Assuming that the left hand side in (17) equals zero, we can apply $(P_2)$ and obtain

\[
\sum_{p \in A^+(q)} y_{pq}^j = 0 = \sum_{s \in A^-(q)} y_{qs}^j.
\]

**Case 2:** Assuming that the left hand side in (17) equals one, there is a unique $\tilde{p} \in A^+(q)$ with $y_{\tilde{p}q}^j = 1$. Hence, the assertion $y_{pq}^j = 0$ holds for all $p \in A^+(q)$ with $p \neq \tilde{p}$. Due to property $(P_1)$, there is a unique $\tilde{s} \in A^-(q)$ with $y_{q\tilde{s}}^j = 1$, and the assertion $y_{qs}^j = 0$ holds for all $s \in A^-(q)$ with $s \neq \tilde{s}$. Thus, we obtain

\[
\sum_{p \in A^+(q)} y_{pq}^j = y_{\tilde{p}q}^j = 1 = y_{qs}^j = \sum_{s \in A^-(q)} y_{qs}^j.
\]
Both cases lead to the equation
\[ \sum_{p \in A^+(q)} y_{pq}^j = \sum_{s \in A^-(q)} y_{qs}^j. \] (18)

Thus, we can finish our computations with
\[ \sum_{p \in A^+(q)} \tilde{x}_{pq} = \sum_{j \in J^*} x_j \sum_{p \in A^+(q)} y_{pq}^j \overset{(18)}{=} \sum_{j \in J^*} x_j \sum_{s \in A^-} y_{qs}^j \]
\[ = \sum_{s \in A^-} \sum_{j \in J^*} y_{qs}^j x_j \overset{(17)}{=} \sum_{s \in A^-} \tilde{x}_{qs}. \]

Hence, all constraints are satisfied. \( \square \)

As above, this theorem also remains true if the continuous relaxations are considered.

**Corollary 6.** Let \( x \) be a feasible solution of the continuous relaxation of the standard model with objective value \( z(x) \). Then there is a feasible solution \( \tilde{x} \) of the continuous relaxation of the reduced arcflow model with the same objective value. In particular,
\[ (z_{Srel})^* \geq (z_{AFrel})^* \]
holds for the optimal values of both models.

These results show that both models provide the same optimal objective value, even in the case where the continuous relaxations are considered. Note that the second part of this observation is not self-evident at all. Actually, there are many fields of discrete optimization (for instance, the traveling salesman problem or Kantorovich-type models in cutting and packing) where different exact modeling frameworks lead to different optimal objective values if the respective continuous relaxations are considered.

### 3 A Model of One-Cut-Type

While in the standard model (complete) patterns are considered, we now take a different approach focusing on each single cutting process that is necessary to split a large stock roll into desired item lengths. Based on this idea the model to be described in this section is denoted as the one-cut model, see [7]. Remarkably, such approaches have never attracted that much scientific interest as, for instance, the arcflow formulation. Indeed, Valério de Carvalho [22] stated:

"To our knowledge, the integer solution of onecut models [...] has never been tried."

Even in the more recent literature [5], this model is only marginally examined which is mainly based on Stadtler’s claim:
“The set of real world cutting stock problems solvable by the one-cut model is only a subset of those which could be tackled by the column generation approach.”

Since, in the end, we aim at comparing the one-cut model and the arcflow model in a preferably convenient way, we change the perspective of this approach slightly. More precisely, note that any single cutting process that splits an item of length $r$ into two items of lengths $p$ and $q$ can likewise be interpreted as adding an item of length $q$ to an item of length $p$. This procedure of adding a single item to another one is denoted as an *one-add*. Thereby, both alternative formulations are somehow based on the same orientation, i.e., from small combined object lengths to larger ones.$^6$

In this section we will present some possibilities to reduce the numbers of variables and constraints, as well as the symmetry in the one-cut model. These principles have mainly been originated from the concepts that were presented for the arcflow formulation in the previous section. As a main contribution, we will also prove the equivalence of the one-cut model and the standard model. In terms of the respective continuous relaxations, this result has not been shown before, and helps to (at least partly) refute the statement of Stadtler [20].

For this purpose, let $D := \{l_1, \ldots, l_m\}$ and $L := \{L_1 < \ldots < L_t\}$ (again with $L_t = L$) denote the set of all item lengths and the set of all maximal pattern lengths, respectively. Furthermore, we define

$$R := \{r \in \{0, 1, \ldots, L - l_m\} \mid \exists a \in \mathbb{Z}_m^+ \setminus \{0, e^1, \ldots, e^m\} : l^\top a = r\}$$

as the set of all intermediate lengths, where $e^i \in \mathbb{Z}_m^+$ denotes the $i$-th unit vector ($i \in I$). Without loss of generality we can assume

- each single adding process to contain at least one demanded item length, and
- that $L \cap D = \emptyset$ holds.

Let $y_{pq} \in \mathbb{Z}_+$ denote the number of items of length $p \in L \cup R$ that are built by adding a demanded item $q \in D$ to an item of length $p - q \in D \cup R$. Note that these intermediate lengths can also represent an additional demanded item, namely if $p - q \in D$ holds. Hence, the set of all feasible one-adds (i.e., the index set for the $y$-variables) is given by

$$M := \{(p, q) \mid p \in L \cup R, p - q \in D \cup R, q \in D, p > q\}.$$

**Remark 7.** We may underline the following observations:

1. Contrary to the skiving stock problem$^7$[26], all three conditions (i.e. $p \in L \cup R$, $p - q \in D \cup R$ and $q \in D$) are important to describe the set of possible one-adds. In particular, discarding $p \in L \cup R$ would lead to items of length larger than $L$ which is not allowed in the cutting context.

---

$^6$For the sake of convenience, the related model is referred to as the one-cut model, too. This “reversed” interpretation of the one-cut model corresponds to the perspective of the bin packing problem, which is strongly related to the one-dimensional cutting stock problem.

$^7$The skiving stock problem (SSP) [26] can be seen as a natural counterpart of the cutting stock problem. Instead of cutting larger items into smaller ones, the objective of the SSP can roughly be described by combining given small items to as many larger ones, specified by some minimum length $L$, as possible.
ii) We emphasize the importance of assumption (iv) from Section 1 for the well-posedness of the one-cut model. Consider, by way of example, the instance \( E = (2, (4, 3), 6, (2, 2)) \) with \( \mathcal{R} = \{3\} \). This leads to the system \( p - q \in \mathcal{D} \cup \mathcal{R} = \{3, 4\}, p \in \mathcal{L} \cup \mathcal{R} = \{3, 4, 6\}, q \in \mathcal{D} \) and \( p > q \) which does not offer any possibility to model the usage of item type \( i = 1 \).

Additionally, we define

\[
N_r := \begin{cases} 
 b_i, & \text{if } r = l_i \in \mathcal{D}, \\
 0, & \text{otherwise},
\end{cases} 
\]

for all \( r \in \mathcal{D} \cup \mathcal{R} \), i.e., the demanded number of items of length \( r \). Then, we obtain the

**One-Cut Model**

\[
z^{OC} = \sum_{k=1}^{t} \sum_{(L_k,q) \in \mathcal{M}} y_{L_k,q} \rightarrow \min
\]

s.t.

\[
\sum_{(q,r) \in \mathcal{M}} y_{q,r} + \sum_{(q+r,q) \in \mathcal{M}} y_{q+r,q} \geq \sum_{(r,q) \in \mathcal{M}} y_{rq} + N_r, \quad r \in \mathcal{D} \cup \mathcal{R},
\]

\[
y_{pq} \in \mathbb{Z}_+, \quad (p,q) \in \mathcal{M}.
\]

The objective function minimizes the number of constructed objects with length in \( \mathcal{L} \). Constraints (20) represent a certain compatibility condition among those one-adds that contain the length \( r \in \mathcal{D} \cup \mathcal{R} \). In particular, this means that at least \( b_i \) items of type \( i \in I \) are used in the whole process (or in the original perspective: that at least \( b_i \) items are cut).

Note that, due to the definition of \( \mathcal{R} \), the approach presented above already contains the concept of potential allocation points. Moreover, we can reduce the numbers of variables and constraints by means of:

i) **Avoiding a double modeling of one-adds:**

Whenever there are two item lengths \( l_i \neq l_j \in \mathcal{D} \) both one-adds \((l_i + l_j, l_i)\) and \((l_i + l_j, l_j)\) lead to the same result. Therefore, we can assume that \( p - q \geq q \) holds in order to avoid a double modeling of the same single adding process.

ii) **Breaking symmetry:**

Similar to the arcflow model, this model also contains several one-add sequences that lead to the same maximal pattern. In analogy to [7], we define an index

\[
\rho(r) := \begin{cases} 
 i, & \text{if } r = l_i \in \mathcal{D}, \\
 \min \{ i \in I \mid r - l_i \in \mathcal{D} \cup \mathcal{R}, i \geq \rho(r - l_i) \}, & \text{otherwise},
\end{cases}
\]

for each \( r \in \mathcal{D} \cup \mathcal{R} \). Therefore, we can reduce the index set of variables to

\[
\mathcal{M}' := \{ (p,q) \mid q = l_i \in \mathcal{D}, p \in \mathcal{L} \cup \mathcal{R}, p - q \in \mathcal{D} \cup \mathcal{R}, p \geq 2q, i \geq \rho(p - q) \}.
\]
We may emphasize that these concepts lead to the

(Reduced) One-Cut Model

\[ z^{OC} = \sum_{k=1}^{t} \sum_{(L_k,q) \in \mathcal{M}'} y_{L_k,q} \rightarrow \min \]

s.t.
\[ \sum_{(q,r) \in \mathcal{M}'} y_{q,r} + \sum_{(q+r,q) \in \mathcal{M}'} y_{q+r,q} \geq \sum_{(r,q) \in \mathcal{M}'} y_{rq} + N_r, \quad r \in \mathcal{D} \cup \mathcal{R}, \quad (22) \]
\[ y_{pq} \in \mathbb{Z}_+, \quad (p,q) \in \mathcal{M}'. \quad (23) \]

Note that this model is also pseudo-polynomial since it contains \( O(mL) \) variables and \( O(m+L) \) constraints. In what follows, we aim at proving the equivalence of this model and the standard model, also in the case where continuous variables are considered. Thereby, we address and answer an important question that has not been dealt with in literature before.

Let
\[ \Omega := \left\{ \{(p_j,q_j)\}_{j=1}^{k} \mid k \in \mathbb{N}, p_1 - q_1 \in \mathcal{D}, p_k \in \mathcal{L}, (p_1,q_1), \ldots, (p_k,q_k) \in \mathcal{M}', \forall \ j \in \{2, \ldots, k\} : p_j - q_j = p_{j-1} \right\} \]
be the set of all sequences of one-adds that lead to an object length of \( \mathcal{L} \), i.e., to the length of a maximal pattern. For simplicity, an element \( \omega \in \Omega \) is called adding sequence. In the following theorem, we exploit the fact that, obviously, each adding sequence can be identified with a maximal pattern.

**Theorem 8.** Let \( y \) be a feasible solution of the reduced one-cut model with objective value \( z(y) \). Then there is a feasible solution \( \tilde{x} \) of the standard model with the same objective value. In particular,

\[ (z^S)^* \leq (z^{OC})^* \]

holds for the optimal values \( (z^S)^* \) and \( (z^{OC})^* \) of both models.

**Proof.** Let \( y \) be a feasible solution of the reduced one-cut model with objective value \( z(y) \). For each \( \omega \in \Omega \) we define a variable \( \sigma^\omega \in \mathbb{Z}_+ \) providing how often this sequence of one-adds is used in the feasible solution \( y \). Due to \( y \in \mathbb{Z}^{[\mathcal{M}']} \), we easily obtain \( \sigma^\omega \in \mathbb{Z}_+ (\omega \in \Omega) \). Consequently, the equality

\[ \sum_{\omega \in \Omega} \sigma^\omega = z(y) \]

holds. For every \( (p,q) \in \mathcal{M}' \) and \( \omega \in \Omega \) we now introduce a decision variable by means of

\[ x_{p,q}^\omega := \begin{cases} 
1, & \text{if } (p,q) \text{ is contained in } \omega, \\
0, & \text{otherwise.} \end{cases} \]

Let
\[ a^\omega := (a_1^\omega, \ldots, a_i^\omega, \ldots, a_m^\omega) \top \in \mathbb{Z}_+^m \]
describe the number of items of type \( i \in I \) being contained in the adding sequence \( \omega \in \Omega \). Because of \( \mathbb{1}^\top a^\omega \in \mathcal{L} \) we obtain \( a^\omega \in \mathcal{P}^*(E) \). Note that the \( i \)-th component of this vector can be determined by

\[
a_i^\omega = \sum_{(p,l_i) \in \mathcal{M}'} x_{p,l_i}^\omega + \sum_{(q+l_i,q) \in \mathcal{M}'} x_{q+l_i,q}^\omega - \sum_{(l_i,q) \in \mathcal{M}'} x_{l_i,q}^\omega.
\]  

(24)

The third sum is necessary since it might be possible to construct an item length \( l_i \) out of smaller item lengths (or in the original perspective: an intermediate length \( r = l_i \) can be further cut to obtain smaller item lengths).

In analogy to the arcflow model, we define an equivalence relation on \( \Omega \) by

\[
\omega_1 \sim \omega_2 \iff a_{\omega_1} = a_{\omega_2}
\]

(25)

for \( \omega_1, \omega_2 \in \Omega \). Based on this, there exists a mapping

\[
\varphi : \Omega / \sim \to \Omega, [\omega] \mapsto \varphi([\omega]) = \omega^*
\]

where \( \omega^* \) represents the unique adding sequence of \( [\omega] \) which is characterized by non-increasing item lengths. Let \( \Omega^* := \text{Im}(\varphi) \), then there is a bijection

\[
\tilde{\kappa} : J^* \to \Omega^*, j \mapsto \tilde{\kappa}(j)
\]

where \( \tilde{\kappa}(j) \) denotes the adding sequence \( \omega^* \in \Omega^* \) with \( a_{\omega^*} = a^j \). In particular, we obtain

\[
a_i^{\tilde{\kappa}(j)} = a_{ij}
\]

(26)

for all \( j \in J^* \) and all \( i \in I \). Setting

\[
\tilde{\sigma}^\omega^* = \sum_{\omega \in [\omega^*]} \sigma^\omega
\]

(27)

for each \( \omega^* \in \Omega^* \), we claim that an appropriate feasible solution of the standard model is given by

\[
\tilde{x}_j := \tilde{\sigma}^{\tilde{\kappa}(j)}
\]

(28)

for all \( j \in J^* \). Indeed, we obtain:

i) Since \( \sigma^\omega \in \mathbb{Z}_+ \ (\omega \in \Omega) \) holds each \( \tilde{x}_j \ (j \in J^*) \) is a nonnegative integer, too.

ii) The objective value results to

\[
\sum_{j \in J^*} \tilde{x}_j \quad (\text{partition}) \equiv \sum_{j \in J^*} \tilde{\sigma}^{\tilde{\kappa}(j)} \ (\tilde{\kappa} \text{ bijective}) \equiv \sum_{\omega^* \in \Omega^*} \tilde{\sigma}^{\omega^*} \sum_{\omega^* \in \Omega^*} \sum_{\omega \in [\omega^*]} \sigma^\omega \equiv \sum_{\omega \in \Omega} \sigma^\omega = z(y).
\]
iii) Constraints (1) are satisfied: indeed, for \( i \in I \), we obtain
\[
\sum_{j \in J^*} a_{ij} \tilde{x}_j = \sum_{j \in J^*} a_{ij} \tilde{\sigma}(j) = \sum_{j \in J^*} a_{i} \bar{\sigma}(j) \quad \text{(\( \bar{\sigma} \) bijective) \( \Rightarrow \)} \sum_{\omega^* \in \Omega^*} a_i^* \tilde{\omega}^* \]
\[
= \sum_{\omega^* \in \Omega^*} \sum_{w \in [\omega^*]} \sigma^\omega \quad \text{[partition]} \sum_{\omega^* \in \Omega^*} \sum_{w \in [\omega^*]} a_i^\omega \tilde{\sigma}^\omega = \sum_{w \in \Omega} \sum_{\omega \in \Omega} a_i^\omega \sigma^\omega \]
\[
= \sum_{\omega \in \Omega} \left( \sum_{(p, l) \in M'} x_{p, l}^\omega \sigma^\omega + \sum_{(q+l, q) \in M'} x_{q+l, q}^\omega \sigma^\omega - \sum_{(l, q) \in M'} x_{l, q}^\omega \sigma^\omega \right) \]
\[
= \sum_{(p, l) \in M'} \sum_{\omega \in \Omega} x_{p, l}^\omega \sigma^\omega + \sum_{(q+l, q) \in M'} \sum_{\omega \in \Omega} x_{q+l, q}^\omega \sigma^\omega - \sum_{(l, q) \in M'} \sum_{\omega \in \Omega} x_{l, q}^\omega \sigma^\omega \]
\[
= \sum_{(p, l) \in M'} y_{p, l} + \sum_{(q+l, q) \in M'} y_{q+l, q} - \sum_{(l, q) \in M'} y_{l, q} \]
\[
\geq N_{l_i} b_i. \]

Observe that for each \((p, q) \in M'\) the equality
\[
\sum_{\omega \in \Omega} x_{p, q}^\omega \sigma^\omega = y_{p, q} \]
has been used in the penultimate line.

This theorem remains true if we consider the continuous relaxations of both models.

**Corollary 9.** Let \( y \) be a feasible solution of the continuous relaxation of the reduced one-cut model with objective value \( z(y) \). Then there is a feasible solution \( \tilde{x} \) of the continuous relaxation of the standard model with the same objective value. In particular,
\[
(z^{S}_{\text{rel}})^* \leq (z^{OC}_{\text{rel}})^* \]
holds for the optimal values of both models.

As we have seen, the methods of the proof are quite similar to Theorem 3. Since this is also true for the second implication we would like to shorten the comments in the following proof slightly.

**Theorem 10.** Let \( x \) be a feasible solution of the standard model with objective value \( z(x) \). Then there is a feasible solution \( \tilde{y} \) of the reduced one-cut model with the same objective value. In particular,
\[
(z^S)^* \geq (z^{OC})^* \]
holds for the optimal values of both models.
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Proof. Let \( x \) be a feasible solution of the standard model with objective value \( z(x) \). For each \((p, q) \in M'\) and all \( j \in J^* \) we define a decision variable by

\[
y_{p,q}^j := \begin{cases} 
1, & \text{if an object of length } q \in D \text{ is added to an object of length } p - q \in D \cup R \text{ in the adding sequence } \tilde{\kappa}(j), \\
0, & \text{otherwise},
\end{cases}
\]

where \( \tilde{\kappa} \) represents the bijection between \( J^* \) and \( \Omega^* \) from Theorem 8. Let \( r \in D \cup R \) and \( j \in J^* \) be given, then there are two important properties:

(A) Let \( r \in R \setminus D \). If there exists \( q \in D \) with \((r, q) \in M'\) and \( y_{r,q}^j = 1\), then there is a unique index \((r + \tilde{q}, \tilde{q}) \in M'\) with \( y_{r+\tilde{q},\tilde{q}}^j = 1\), i.e., the equality \( y_{r+q',q'}^j = 0\) holds for all \((r' + q', q') \in M'\) with \( q' \neq \tilde{q}\).

(B) If for all \( q \in D \) with \((r, q) \in M'\) the assertion \( y_{r,q}^j = 0\) holds, then also \( y_{r+\tilde{q},\tilde{q}}^j = 0\) is true for all \((r + \tilde{q}, \tilde{q}) \in M'\).

Defining

\[
\tilde{y}_{p,q} := \sum_{j \in J^*} y_{p,q}^j x_j
\]

for every \((p, q) \in M'\), we claim that we found an appropriate feasible solution of the reduced one-cut model. Indeed, we obtain:

i) Since \( x_j \in \mathbb{Z}_+ \) \((j \in J^*)\) holds the nonnegative integrality of (30) follows from (29).

ii) The objective value results to

\[
\sum_{k=1}^{t} \sum_{(L_k,q) \in M'} \tilde{y}_{L_k,q}^{j} = \sum_{k=1}^{t} \sum_{(L_k,q) \in M'} \sum_{j \in J^*} y_{L_k,q}^j x_j = \sum_{j \in J^*} \sum_{k=1}^{t} \sum_{(L_k,q) \in M'} y_{L_k,q}^j x_j = \sum_{j \in J^*} x_j = z(x).
\]

Note that each adding sequence has to end with an uniquely determined element of \( \mathcal{L} \) such that the assertion related to the curly bracket is justified.

iii) Let \( r = l_i \in D \) be given, then

\[
\sum_{(r+q',q') \in M'} y_{r+q',q'}^j + \sum_{(q,r) \in M'} y_{q,r}^j - \sum_{(r,q) \in M'} y_{r,q}^j = a_{ij}
\]

holds for all \( j \in J^* \) since all objects of length \( l_i \) in \( \tilde{\kappa}(j) \) that have not been constructed out of smaller item lengths have to appear in the maximal pattern \( a^j \), cf. (24).
Hence, we obtain
\[
\sum_{(r+q,q) \in M'} \tilde{y}_{r+q,q} + \sum_{(q,r) \in M'} \tilde{y}_{q,r} - \sum_{(r,q) \in M'} \tilde{y}_{r,q} \tag{30}
\]
\[
= \sum_{j \in J^*} x_j \left( \sum_{(r+q,q) \in M'} y_{r+q,q}^j + \sum_{(q,r) \in M'} y_{q,r}^j - \sum_{(r,q) \in M'} y_{r,q}^j \right) \tag{31}
\]
\[
= \sum_{j \in J^*} a_{ij} x_j \geq b_i N_r. \tag{19}
\]

iv) Let \( r \in \mathcal{R} \setminus \mathcal{D} \) be a “real” intermediate length. Then, \( M' \) does not contain any element of type \((q,r)\) since the second component of such a pair has to belong to \( \mathcal{D} \). It follows that the assertion
\[
\sum_{(q,r) \in M'} y_{q,r}^j = 0 \tag{32}
\]
is true for all \( j \in J^* \). Furthermore, the relation
\[
\sum_{(r+q,q) \in M'} y_{r+q,q}^j - \sum_{(r,q) \in M'} y_{r,q}^j = 0 \tag{33}
\]
holds for every \( j \in J^* \) since the right sum contains at most one nonzero element. If this sum equals zero, assertion (B) leads to (33); otherwise we can apply (A) obtaining the same result.

Altogether, it follows that
\[
\sum_{(r+q,q) \in M'} \tilde{y}_{r+q,q} + \sum_{(q,r) \in M'} \tilde{y}_{q,r} - \sum_{(r,q) \in M'} \tilde{y}_{r,q} \tag{30}
\]
\[
= \sum_{j \in J^*} x_j \left( \sum_{(r+q,q) \in M'} y_{r+q,q}^j + \sum_{(q,r) \in M'} y_{q,r}^j - \sum_{(r,q) \in M'} y_{r,q}^j \right) \tag{32}
\]
\[
= \sum_{j \in J^*} x_j \left( \sum_{(r+q,q) \in M'} y_{r+q,q}^j - \sum_{(r,q) \in M'} y_{r,q}^j \right) \tag{33}
\]
\[
= \sum_{j \in J^*} x_j \cdot 0 = 0 \tag{19}
\]
which completes the proof. \( \square \)

This theorem remains true if we focus on the continuous relaxations of both models.
Corollary 11. Let $x$ be a feasible solution of the continuous relaxation of the standard model with objective value $z(x)$. Then there is a feasible solution $\tilde{y}$ of the continuous relaxation of the reduced one-cut model with the same objective value. In particular,

$$(z_{rel}^S)^* \geq (z_{rel}^{OC})^*$$

holds for the optimal values of both models.

Hence, we have proved the equivalence of all three approaches, i.e., the standard, the (reduced) arcflow and the (reduced) one-cut model. Remarkably, this equivalence also holds in the continuous case. In particular, the alternative models (i.e., the arcflow model and the one-cut model) can be solved by standard software like CPLEX also offering a possibility to investigate the gap [17, 18] of the cutting stock problem.

4 A Comparison of the Arcflow Model and the One-Cut Model

In the last sections, we noticed a certain relationship between the two alternative approaches. Both of them avoid the modeling of (entire) patterns and only consider the position or order of the single items. Obviously, each one-add $(p, q) \in M'$ can also be interpreted as the positioning of an item with length $q$ at the vertex $p - q \in V'$ in the graph $G'$, i.e., as an arc $(p - q, p) \in E'$. Thus, in a certain extent, both alternative models share a common structure of variables. We therefore aim at investigating which of these models is expected to require a lower computational effort. To this end, we particularly focus on the two criteria of the number of variables and the number of constraints.

Proposition 12. The assertion

$$n_{con}^{AF} - n_{con}^{OC} = m$$

holds for the numbers of constraints in the alternative models.

Proof. Note that

$$V' \setminus (L \cup \{0\}) = D \cup R$$

is true since both sets are equal to $\{ w \in \{l_m, \ldots, L - l_m\} \, | \, \exists a \in \mathbb{Z}_m^+: I^\top a = w \}$. Hence, it follows that

$$n_{con}^{AF} - n_{con}^{OC} = (|V' \setminus (L \cup \{0\})| + m) - |D \cup R| + m$$

which proves the proposition.

The proof of a corresponding result for the number of variables is much more complex. Therefore, we split it into different steps starting with the following relation between the indices $\mu$ and $\rho$.

Lemma 13. The assertion

$$\mu(r) = \rho(r)$$

holds for all $r \in D \cup R$. 
Proof. Let \( r \in D \cup R \) be given, then \( \mu \) is well defined due to (34). At first, we consider the case \( r = l_i \in D \), where \( \rho(r) = i \) holds by definition. We prove that \( \mu(r) = i \) is true, too.

i) The inequality \( \mu(r) \geq i \) holds: For \( i = 1 \) this assertion is clear since \( \mu(r) \in I \) has to be satisfied. For \( i \geq 2 \) the inequality \( r - l_{i-1} < 0 \) holds implying that \( r - l_{i-1} \notin \mathcal{V}' \) which leads to \( \mu(r) \geq i \).

ii) The inequality \( \mu(r) \leq i \) holds: Obviously, \( r - l_i = 0 \in \mathcal{V}' \) and \( i \geq \mu(r - l_i) = \mu(0) = 0 \) are true. From the minimization in (7), we can conclude \( \mu(r) \leq i \).

Now let \( r \in R \setminus D \) be given. Then, the assertion is obviously true since both definitions are equal, cf. (34).

In order to exploit the above mentioned relationship between the index sets of the variables within the alternative models, we consider the mapping

\[
f : \mathcal{M}' \to \mathcal{E}', \quad (p,q) \mapsto (p - q, p).
\]

It can easily be verified that \( f \) is well-defined and injective.

**Lemma 14.** The inequality

\[
n_{\text{var}}^{AF} - n_{\text{var}}^{OC} \geq m
\]

holds for the numbers of variables in the alternative models.

**Proof.** We show that for each \( i \in I \) the pair \((0, l_i) \in \mathcal{E}'\) has no preimage under \( f \), and, hence, the assertion

\[
|\mathcal{E}'| - |\text{Im}(f)| \geq m
\]

is true. For the sake of contradiction, we assume an index \( i \in I \) and an arc \((p, q) \in \mathcal{E}'\) to satisfy \( f(p, q) = (0, l_i) \). Then, the second component of this equality implies \( p = l_i \). Due to \( p - q = 0 \), the equality \( q = l_i \) follows immediately. But then, \( p \geq 2q \) is not possible which gives the contradiction. Hence, we obtain

\[
n_{\text{var}}^{AF} - n_{\text{var}}^{OC} = |\mathcal{E}'| - |\mathcal{M}'| = |\mathcal{E}'| - |\text{Im}(f)| \geq m
\]

since \( f \) is a bijection between \( \mathcal{M}' \) and \( \text{Im}(f) \).

The reverse inequality needs the following preliminary result.

**Lemma 15.** The inequality

\[
2\tilde{p} - \tilde{q} \geq 0
\]

holds for all \((\tilde{p}, \tilde{q}) \in \mathcal{E}'\) with \( \tilde{p} \neq 0 \).

**Proof.** Let \((\tilde{p}, \tilde{q}) \in \mathcal{E}'\) with \( \tilde{p} \neq 0 \) be given. We assume the statement to be wrong, i.e., \( 2\tilde{p} - \tilde{q} < 0 \) or equivalently \( \tilde{p} < \tilde{q} - \tilde{p} \).

\[
\tilde{p} < \tilde{q} - \tilde{p}.
\]  

(36)
i) Since \( (\tilde{p}, \tilde{q}) \in \mathcal{E}' \) holds there exists \( i^* \in I \) with \( \tilde{q} - \tilde{p} = l_{i^*} \). Let \( j^* := \mu(\tilde{p}) \), then we obtain
\[
i^* \geq j^* \tag{37}
\]
due to the definition of \( \mathcal{E}' \).

ii) By reason of \((36)\) we note that \( l_{i^*} > \tilde{p} \) holds. In particular, \( l_{i^*} \) is larger than all objects appearing in paths from \( v_0 = 0 \) to \( \tilde{p} \) in the graph \( \mathcal{G}' = (\mathcal{V}', \mathcal{E}') \). Let \( \Gamma(\tilde{p}) \) denote the set of these paths, then we can certainly state \( \Gamma(\tilde{p}) \neq \emptyset \) due to \( \tilde{p} \in \mathcal{V}' \) and \( \tilde{p} \neq 0 \). Since \( \mu(\tilde{p}) = j^* \) holds there has to be a path in \( \Gamma(\tilde{p}) \) containing an item of index \( j^* \). This leads to \( l_{i^*} > l_{j^*} \), contradicting \((37)\).

Hence, the initial assumption was wrong and the lemma is proved. \( \square \)

Now we are able to prove the second inequality.

**Lemma 16.** The inequality
\[
n_{\text{var}}^{AF} - n_{\text{var}}^{OC} \leq m
\]
holds for the numbers of variables of the alternative models.

**Proof.** Let \( (\tilde{p}, \tilde{q}) \in \mathcal{E}' \) with \( \tilde{p} \neq 0 \) be given. We show that \( |\mathcal{E}'| - |\text{Im}(f)| \leq m \) is true proving \( (\tilde{p}, \tilde{q}) \in \text{Im}(f) \) by means of indicating a corresponding preimage. To this end, observe that
\[
f(\tilde{q}, \tilde{q} - \tilde{p}) = (\tilde{p}, \tilde{q})
\]
holds obviously. Defining \( p := \tilde{q} \) and \( q := \tilde{q} - \tilde{p} \), only \( (p, q) \in \mathcal{M}' \) remains to prove. Indeed, we obtain:

i) From \( q = \tilde{q} - \tilde{p} = l_i \in \{l_1, \ldots, l_m\} \), the assertion \( q \in \mathcal{D} \) follows directly.

ii) First of all, observe that \( p - q = \tilde{q} - (\tilde{q} - \tilde{p}) = \tilde{p} \in \mathcal{V}' \) holds. From the definition of \( \mathcal{E}' \), we obtain \( \tilde{p} < L \) leading to \( \tilde{p} \notin \mathcal{L} \). According to \( \tilde{p} \neq 0 \), we conclude
\[
p - q = \tilde{p} \in \mathcal{I} \quad \mathcal{D} \cup \mathcal{R}.
\]

iii) Due to \( \tilde{p} \neq 0 \) we have \( \tilde{p} \in \mathcal{V} \setminus \{0\} \), i.e., there is a vector \( a \in \mathbb{Z}_+^n \setminus \{0\} \) with \( l^\top a = \tilde{p} \).

Thus, we can conclude that \( p = \tilde{q} = \tilde{p} + (\tilde{q} - \tilde{p}) \in \mathcal{L} \cup \mathcal{R} \) holds.

iv) By Lemma \ref{lem:13} we obtain \( 2\tilde{p} - \tilde{q} \geq 0 \), such that \( p \geq 2q \) holds.

v) Due to \( \tilde{p} \in \mathcal{D} \cup \mathcal{R} \) (cf. (ii)), Lemma \ref{lem:13} is applicable and leads to
\[
i \geq \mu(\tilde{p}) \implies \rho(\tilde{p}) = \rho(\tilde{q} - (\tilde{q} - \tilde{p})) = \rho(p - q).
\]

Altogether, we obtain \( (p, q) \in \mathcal{M}' \) and \( (\tilde{p}, \tilde{q}) \in \text{Im}(f) \) entailing
\[
n_{\text{var}}^{AF} - n_{\text{var}}^{OC} = |\mathcal{E}'| - |\mathcal{M}'| = |\mathcal{E}'| - |\text{Im}(f)| \leq m
\]
which completes the proof. \( \square \)

Hence, we have shown that the one-cut model differs from the arcflow model by exactly \( m \) fewer variables and constraints. In general, especially for practical meaningful instances, this number is “small” in contrast to both the total numbers of variables \( \mathcal{O}(mL) \) and restrictions \( \mathcal{O}(m + L) \). We therefore expect the average calculation times of the one-cut model to be smaller than those ones of the arcflow model, but not differing that much among one another. This behavior will be investigated numerically in the next section.
5 Computational Results

In order to compare the performances of the presented models we implemented them in MATLAB R2015b and used its CPLEX-interface (version 12.6.1) to solve all optimization problems. Thereby, our computational environment was given by an Intel Core i5-2450M CPU with 2.50GHz and 6GB RAM. We may emphasize that the numerical results presented in this section shall be considered as an auxiliary tool to exemplary illustrate the behavior of both models for different input-parameters. Indeed, since our manuscript is rather theoretically oriented, we do not intend to present an exhaustive computational study regarding all possible aspects connected with the cutting stock problem.

In a first series, we randomly generated 50 instances each for the given pairs $(m, L)$. Thereby, the lengths $l_i$ and demands $b_i$ were chosen from uniformly distributed integer numbers in $[L/4, 3L/4]$ and $[1, 50]$, respectively. Hence, we are dealing with rather simple instances. The abbreviations in Tab. 1 and all upcoming tables shall be interpreted as follows:

- $t$ represents the average (pure) solution time (in seconds) for the corresponding ILP,
- $t_{rel}$ represents the average (pure) solution time (in seconds) for the corresponding LP relaxation,
- $t_{mod}$ denotes the average time (in seconds) for precalculations (e.g. graph construction, reduction techniques, matrix definitions),
- $n_{var}$ denotes the average number of variables,
- $n_{con}$ denotes the average number of constraints,
- $n_{iter}$ denotes the average number of iterations within the solution technique applied by CPLEX.

Since large items are allowed and very small items are forbidden in this first series of test instances, the complexity of the resulting optimization problems is rather low. Hence, as to be seen in Tab. 1, both approaches nearly need the same time in each scenario (regarding $t$, $t_{rel}$ and $t_{mod}$). Consequently, the predicted numerical behavior cannot be noticed for these small instances. Nevertheless, it can be observed that the difference between the numbers of constraints or variables is always equal to $m$ as proved in the theoretical part of this paper. In order to see some larger differences between the computation times, we used uniformly distributed integer lengths $l_i \in [L/10, L/2]$ (and again $b_i \in [1, 50]$) in our second experiment, see Tab. 2. Note that the modified interval for the item lengths leads to a larger variety of possible combinations between the single items, in general. Thereby, the number of potential allocation points, as well as the number of arcs and one-adds (or one-cuts in the original orientation) is likely to increase which entails more variables and constraints. Altogether, due to these observations, larger computation times can be expected.

In this second set of instances, the differences between the arcflow model and the one-cut model can be seen a little bit more clearly. In most cases, the average times are (slightly)
Table 1: computational results for uniformly distributed integers $l_i \in [L/4, 3L/4]$

<table>
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<th></th>
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<td>0.01</td>
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<td>0.02</td>
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<td>0.02</td>
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<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
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</tr>
<tr>
<td></td>
<td>$t_{mod}$</td>
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<td>0.01</td>
<td>0.02</td>
<td>0.01</td>
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Table 2: computational results for uniformly distributed integers \( l_i \in [L/10, L/2] \)

<table>
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<td>0.10</td>
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<td>57.44</td>
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<td>0.19</td>
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<td>0.03</td>
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<td>( t_{mod} )</td>
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<td>0.02</td>
<td>0.08</td>
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<td>0.23</td>
</tr>
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<td>81.88</td>
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</table>
smaller for the one-cut approach, especially regarding the ILP formulations and the pre-calculations. Concerning the last point it should be noted that the arcflow model requires an additional system matrix for satisfying the order demands, whereas the efforts for implementing conditions (5) or (22) can be expected to be nearly the same for the respective approaches.

Actually, due to the smaller item sizes, these instances have proved to be harder than the previous ones, particularly in terms of their numbers of variables and constraints. Furthermore, observe that, in general, the average computation times of both models are the higher the larger the parameters \( m \) and \( L \) are. This behavior is reasonable since both of them possess \( O(mL) \) variables and \( O(m + L) \) constraints which, in most cases, lead to additional computational expenses if \( m \) or \( L \) increases. Concerning this matter, we also emphasize the huge effects that are related to the presented reduction principles. Consider, by way of example, the set of instances characterized by \((m, L) = (30, 300)\): in theory, we had to expect roughly \( m \cdot L = 9000 \) variables and \( m + L = 330 \) constraints in both modeling frameworks. As to be seen, particularly the number of variables is significantly smaller than this theoretical upper bound. Hence, the proposed reductions effect high economies as regards the complexity of the models since they considerably lessen the numbers of variables and restrictions. This fact can be observed even more clearly when considering instances with larger input-data.

To this end, a third experiment deals with very large problems up to the combination \((m, L) = (100, 5000)\). Thereby, we randomly selected uniformly distributed integer lengths \( l_i \in [L/10, 3L/4] \) to allow rather small items as well as rather large items. Again \( b_i \) was randomly chosen from \([1, 50]\) as in the previous cases. The corresponding results can be found in Tab. 3.

Especially for the very large instances (e.g. for \((m, L) = (100, 5000)\)) the one-cut approach turns out to possess smaller computation and precalculation times. Surprisingly, this observation does not hold for the continuous relaxations in every case. Nevertheless, whenever the arcflow approach requires less time regarding the relaxed models, the corresponding time difference is really small so that the performance of both formulations is nearly equal. Furthermore, our results show that varying parameter \( m \) has significantly larger influences (compared to modifying parameter \( L \)) on the obtained experimental data. Actually, due to \( O(mL) \) variables and \( O(m + L) \) constraints, both of them should be expected to nearly possess the same relevance for the complexity of the problem. However, note that especially in terms of precalculations, there are some steps whose expenses may strongly depend on \( m \) (e.g. the recursive computation of \( \mu \) or \( \rho \)). Furthermore, a larger number of different item lengths can increase the likelihood that more symmetries (as described in Remark 2) remain in the reduced models; whereas the parameter \( L \) has almost no influence on that aspect.

Concerning the number of iterations\(^8\) for solving the corresponding discrete optimization

---

\(^8\)Note that the number of CPLEX iterations particularly contains different procedures from the preprocessing part, such that it is not equivalent with the number of visited nodes in the branch-and-bound tree. In fact, for almost all instances, the optimal solution is found at the root node. This observation is mainly due to the fact that our randomly generated instances always possessed the integer round-up property (IRUP), i.e., the optimal objective value of the integer problem is already known after having solved the LP relaxation for the root node.
Table 3: computational results for uniformly distributed integers $l_i \in [L/10, 3L/4]$

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<th>$m = 100$</th>
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<td>0.08</td>
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<td>0.34</td>
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problems by means of the branch-and-bound technique applied by CPLEX, we obtain quite heterogeneous simulation results. In most cases, the number of iterations increases when the problem parameters become larger, but also the opposite effect can be noticed (for instance in the row belonging to $L = 2000$). Moreover, also a consideration of the two investigated modeling frameworks does not lead to a clear numerical statement when comparing their iteration numbers. This variable behavior might be caused by the fact that CPLEX chooses an appropriate starting point for the iterations and applies some further reductions internally that may strongly depend on the particular structure of the given problem (which is mainly determined by the input data and the constraints). However, since these procedures are contained within the unknown CPLEX solving strategy, a more detailed analysis of this effect within the scope of our manuscript is not possible.

Altogether, our implementations have shown that, for instances of moderate sizes, both models can be considered as equivalent from a computational point of view. However, the one-cut approach has turned out to possess (slight) advantages when the complexity of the models increases.

**Remark 17.** Note that the method of randomly-generated instances can be generalized to so-called problem generators, see [8] for an appropriate approach. As regards the generation of the item lengths we are working quite close to the construction proposed in that paper (especially since we also use lower and upper bounds $\nu_1, \nu_2 \in (0, 1)$ for the relative item sizes $l_i/L$). However, a main difference is given by the fact that we also kept the values of $b_i$ fully randomly. In particular, we did not fix $\sum_{i=1}^{m} b_i$ as it was done in [8].

### 6 Conclusions

In this manuscript we considered two alternative formulations for the one-dimensional cutting stock problem, the arcflow model and the one-cut model. We reviewed and presented possible reduction techniques for both of these approaches and investigated their influence in numerical simulations. As a main contribution we showed the equivalence between the standard pattern-based model of Gilmore and Gomory and the (reduced) one-cut formulation. Remarkably, this result remains true for the respective continuous relaxations. This equivalence has not been noticed before in literature; thereby, it answers an important open question in the field of one-dimensional cutting. Moreover, we dealt with a theoretical comparison between the arcflow model and the one-cut model and verified the obtained results in computational simulations based on randomly generated instances. Altogether, this work contributes to consider one-cut approaches as a competitive alternative for solving cutting stock problems.

### References


A Comparative Study of the Arcflow Model and the One-Cut Model for the 1d-CSP


