Annotated Bibliographies in Combinatorial Optimization,

Cutting and Packing

Harald Dyckhoff
Aachen University of Technology (RWTH)

Guntram Scheithauer
Dresden University of Technology

Johannes Terno
Dresden University of Technology

Contents

1. History and Development of C&P
2. Knapsack Packing
   2.1. Knapsack problems
   2.2. Related one-dimensional C&P problems
   2.3. Two-dimensional rectangular C&P problems
3. Pallet Loading
4. Bin Packing
5. Stock Cutting
   5.1. One-dimensional cutting stock problems
   5.2. Two-dimensional cutting stock problems
   5.3. Sequencing and assortment problems
6. General Two-dimensional C&P
   6.1. Strip packing and layout problems
   6.2. Non-rectangular C&P problems
7. Three Dimensional C&P
8. Related Topics
9. Software

The field of Cutting and Packing (C&P) encompasses a variety of multiple connected, geometrical as well as combinatorial problems, models and algorithms on theory and practice. The main objective is the efficient arrangement of geometrically described pieces in larger domains. C&P are hereby two equivalent ways to characterize a given or desired arrangement. The following list of problems strongly related to C&P illustrates this topic: cutting
stock and trim loss problem, bin packing, strip packing, knapsack problem; vehicle loading, pallet loading, container loading, design, layout, nesting and partitioning problem; capital budgeting, change making, line balancing, memory allocation and multiprocessor scheduling problem.

C&P problems are stated and treated in different disciplines such as Management Science, Engineering Science, Information and Computer Sciences, Mathematics as well as Operational Research. They appear in many practical branches such as the glass, steel, wood and paper industry, clothing and leather industry, cargo loading and logistics.

Many heuristic and exact algorithms and solution methods have been developed since the mid fifties of this century. Such approaches concern the application of branch-and-bound, dynamic programming, simulation, dialog techniques, revised simplex method, priority rules, enumeration and greedy techniques as well as various heuristics.

In drawing up this bibliography, we have concentrated on publications that appeared in 1985 or later (with some exceptions). We have exercised some judgement in determining which publications to include. If any reader feels we have overlooked an important contribution, we would be pleased to hear from her or him. For the literature not included we refer to the books and surveys listed in §1, especially to the book of Dyckhoff/Finke (1992).

In this paper we have included more than 150 annotated references. A BiBTeX data base containing more than 1200 entries related to C&P can be obtained from the authors.

1 History and Development of C&P

In the following some milestones in the development of C&P are considered.


These are the first mathematical articles in the field of cutting and packing.


For the first time, the articles present techniques which can be used in practice for problems of medium size. The authors describe the meanwhile classical linear programming approach to one-dimensional cutting stock problems (successive use of new cutting patterns via column generation in the revised simplex method). Furthermore more-dimensional guillotine patterns are introduced and optimal patterns are determined using dynamic
programming. The papers also show the importance of the **knapsack problem** as a sub-

problem occurring in many cutting stock problems.


For the first time, Herz uses integral combinations of piece lengths and widths (so-called **raster points**) to normalize packing patterns which leads to an essential reduction of computational amount. The **symmetrization** and the use of **bounds** are also considered.


The authors give a branch-and-bound approach for the so-called **constrained guillotine cutting problem**. Constrained means that the number of times a piece is contained in a cutting pattern is restricted. They consider an enumeration of the possible cuts that can be made on the stock rectangle. Upper bounds are calculated by means of transportation routine and dynamic programming routine.


Wang discusses the generation of guillotine patterns by successively combining of pieces together.


These books contain some of the work done in the former USSR published in Russian. Problems of arranging rectangular and non-rectangular pieces on strips and other domains are considered theoretically as well as algorithmically and for applications.


The authors collect various cutting problems and algorithms for practical solution. For solving rectangular cutting stock problems efficient dynamic programming recursions are given, introducing the so-called **reduced raster point sets**.

An important step to support the international research on C&P was the creation of the informal **Special Interest Group on Cutting and Packing (SICUP)** by H. Dyckhoff and G. Wäscher in 1988. SICUP has an own bulletin published approximately twice a year. Special streams of papers on C&P have been initiated by SICUP at international conferences.

Dyckhoff presents a typology for C&P problems which is based on the investigation of characteristics of their geometrical and logical structure and of their appearance in reality:

- **Dimensionality:** One-, two-, three- or n-dimensional problems.
- **Type of assignment:** An essential characterization results from whether a complete stock of large objects or small items must be assigned or only a suitable selection.
- **Characteristics of large objects and small items:** e.g. the type of quantity measurement, figure (shape), assortment, availability.
- **Pattern restrictions:** Technological conditions of the C&P process like guillotine cutting, stage number, frequency of patterns or the number of items, etc.
- **Objectives:** Objectives are the minimization of material or space, trim loss minimization, value maximization, etc.


The book contains approximately 700 entries in the field of C&P. The authors develop a characterization scheme similar to the above mentioned and give a systematic survey of the existing literature.


The research on C&P is focused in these surveys.

The rapid expansion of literature related to C&P leads to the edition of special issues of some journals:


2 Knapsack Packing

In knapsack packing a selection from a large supply of items of heterogeneous figures must be assigned to only a limited stock of objects. Usually one object is considered. Modifications arise if the number of pieces of a certain type can be bounded, the objects may contain some forbidden regions for some of the items, or the sizes of the pieces may vary in given tolerances, etc.

2.1 Knapsack problems


A unified approach and a summary of the most important results concerning the exact methods for solving the (binary) knapsack problem and its generalizations are given.


The procedure developed uses a forward state generation scheme of the dynamic programming along with a facility to eliminate the nonpromising states by calculating an upper bound on the best return that can be achieved with the remaining variables.


The book gives a comprehensive compilation of problems related to the Knapsack Problem and solution methods. It contains chapters on Knapsack, Bounded knapsack, Subset-sum, Change-making, Multiple knapsack, Generalized assignment and Bin packing problem. 13 codes are contained on an assigned floppy disc.


The NP-complete separation problem for the knapsack polyhedron is formulated as a side-constrained network flow problem with a pseudo-polynomial number of vertices and edges.


A survey of worst case performance of greedy algorithms for the two model problems Knapsack and Maximum travelling salesman is given in the paper.


Conditions for a knapsack to be greedy solvable are studied, and necessary and sufficient conditions, verifiable in polynomial time, are presented. Furthermore, a family of
matroidal relaxations for the 0–1 knapsack problem is investigated.


The algorithm consists of a lexicographic search in which, for different possible values of an integer $k$, the best solution which inserts exactly $k$ items in the knapsack is determined. The algorithm efficiently solves strongly correlated instances.


The algorithm provides sharp lower and upper bounds on the optimal value, and also a tighter equivalent representation by reducing the continuous feasible set and by eliminating constraints and variables.


The article surveys several methods that can be used to solve integer knapsack problems on a variety of parallel computing architectures.


Structural properties of the knapsack problem are investigated in these papers.


A new fully polynomial approximation scheme is presented.


This paper deals with parametric knapsack problems where the costs resp. weights are replaced by linear functions depending on a parameter. The aim is to find the smallest parameter such that the optimal solution value of the knapsack problem is equal to a pre-specified solution value.


Upper bounds are presented, obtained from the mathematical model of the problem by adding valid inequalities on the cardinality of an optimal solution, and relaxing it in a Lagrangian fashion.
2.2 Related one-dimensional C&P problems


These papers deal with practical applications in which additional constraints and/or generalized objectives arise when defects of the stock material or other quality conditions have to be regarded.


Here case studies are given.

2.3 Two-dimensional rectangular C&P problems

In these problems, small rectangles (pieces) of various sizes must be assigned to a choice of large rectangles (objects). Mostly, one object is considered. In case of orthogonal cutting or packing the pieces have to be arranged parallel to the edges of the objects. A guillotine cut means a straight line from an edge of the object to the opposite one, parallel to the others. A problem is said to be constrained if the number of pieces of a certain type is restricted additionally in the pattern looked for.


New heuristics are given for the constraint guillotine cutting of a rectangle. In the second paper an $O(n^3)$ heuristic is presented.


These papers give exact approaches for solving the constraint guillotine cutting problem.


Only guillotine cuts with respect to the stock rectangle are allowed in order to get feasible patterns.


A dynamic programming approach for three-stage guillotine cutting of rectangles is developed regarding non-usable areas on the stock material.


A Lagrangian relaxation of a zero-one integer programming formulation of the problem is developed and used as a bound in a tree search procedure. Subgradient optimization is used to optimize this bound. Problem reduction tests derived from both the original problem and the Lagrangian relaxation are given.


A new graph-theoretic approach is considered.


The packing of small identical rectangles within a strip is considered allowing a non-orthogonal position of the items.


An effective heuristic for solving two-dimensional cutting problems is presented. The algorithm selects a subset of generated strips by solving a sequence of one-dimensional knapsack problems.

### 3 Pallet Loading

The pallet loading problem (PLP) looks for an optimal arrangement of a selection of identical small items to one large object. In general the optimal patterns have typical block or parquet structure. In some cases additional criteria are applied to determine a suitable pattern.

Consideration is given to the development of loading patterns into pallet stacks, and criteria which might be applied to determine the sensibility of these stacks for storage and transportation. A technique is developed which allows the stability and clampability to be tested, and this is applied to layouts produced by an algorithm.


A new LP-based upper bound is proposed for the PLP which consists of an optimal packing of identical small rectangles on a pallet.


A branch-and-bound algorithm is developed which is based on a transformation of the PLP to the independent set problem.


Based on a pallet chart the paper includes sensitivity analysis with respect to changes of box and pallet dimensions.


The upper bound for an instance $E$ of the PLP is based on the minimization of the area bound with respect to all instances equivalent to $E$.


This paper provides a survey of recent research relating to considerations in pallet packing, the models and solution procedures used in pallet packing, and implementation of these approaches in palletization stations. (47Refs.)


A branch-and-bound algorithm is proposed.


A polynomial time algorithm is presented having complexity $O(\Theta\log^2 w)$ where $\Theta = \log_2 \max\{L, W, l, w\}$ and $L \times W$ and $l \times w$ are the sizes of the pallet and boxes, respectively.


These Ph. D. theses contain surveys of optimization algorithms and extensive computational results.


A new upper bound method is discussed. It is based on a set of structural constraints for a solution which realites the assumed upper bound.


This covers the application of a pallet chart to practical problems where product volume might be variable.


The new heuristic is based on structure investigations and it dominates all heuristics known so far. Moreover, the G4-heuristic produces in any case optimal patterns if not more than 50 items can be packed.

4 Bin Packing

In the bin packing problem numerous small, heterogeneously-shaped (one–, two– or three–dimensional) items have to be assigned to a selection of objects. In many cases a successive assignment of the items to the objects is an essential restriction of the problem.


The presented shelf heuristic is an on-line algorithm. The expected performance is analyzed and the choice of the shelf sizes is discussed.


The book examines techniques useful in the probabilistic analysis of algorithms, focusing on applications to bin packing and partitioning. Since the book concentrates as much on techniques as on results it is a useful introduction to probabilistic analysis even for those readers with interests in other problem areas.


10
The cost of a bin is a concave function in the number of items in the bin. The objective is to store the items in such a way that total cost is minimized.


Heuristics for the guillotine bin packing problem are proposed.


Lower bounds for the asymptotic worst case ratio of on-line algorithms for different kind of bin packing problems are discussed.


A *level-oriented* algorithm is described and analyzed, called *Reverse-Fit*, for packing rectangles into a unit-width, infinite-height bin so as to minimize the total height of the packing.


New worst-case results are presented for a number of classical heuristics. It is shown that the first-fit and best-fit heuristics have an absolute performance ratio of no more than 1.75, and first-fit decreasing and best-fit decreasing heuristics have an absolute performance ratio of 1.5.


A known lower bound is analyzed and its worst case performance is determined. Moreover, new lower bounds are proposed which are used within a branch-and-bound algorithm. Results of numerical tests with instances involving up to 120 pieces are presented.

## 5 Stock Cutting

*Stock cutting* denotes those C&P problems in which many items of only several distinct shapes must be assigned to a selection of objects. Contrary to bin packing problems these groups of identical items allow in general the multiple use of certain patterns in the solution.
5.1 One-dimensional cutting stock problems


The "rounding" algorithm performs the task of producing an integer solution from the (fractional) LP solution while at the same time reducing the number of distinct cutting patterns and removing, where possible, any pattern with small quantity.


The so-called complete cut and one-cut models are compared.


Problems and solution procedures are considered in which not only trim loss is regarded.


The primary objective is to minimize the trim loss in a given piece of metal work requiring metal sections of various lengths. The secondary objective is to organize the cutting so that the maximum quantity of leftovers is accumulated in the last bar(s).


In the steel industry when a finished structural shape exits the mill upon which it was produced it is cut into customer order lengths. The actual length of the finished beam bar may not be known precisely until immediately before cutting.

In the following papers the difference between the optimal values of the integer and the corresponding LP-relaxation problem is investigated. If this difference is less than 2 then the so-called Modified integer round-up property holds.


A problem generator for the standard one-dimensional cutting stock problem is developed.


The authors compare various heuristic approaches and show that two of them are clearly superior to the others as well as solve almost any of 4,000 randomly generated test problems to an optimum.

5.2 Two-dimensional cutting stock problems


The adaptations considered relate to initial pattern generation, multiple solutions per knapsack problem, explicit valuation of undersupply and oversupply, and waste having a value.


A problem formulation in a fuzzy environment is presented which addresses a number of practical aspects also of interest besides the waste minimization.


Case studies are presented.


Different cutting and related problems are considered like forming problems (closed packing problems) and problems of cutting planning. For solving these planning problems, linear or integer programming is used.


An LP-solution is rounded off according to a special procedure and some further improvements are given via a branch-and-bound approach.

The authors present an algorithm to obtain optimal integer solutions.


The maximum difference between the optimal values of the integer and the corresponding LP-relaxation problem is investigated for two-dimensional cutting stock problems.


An LP model for a two-stage cutting stock problem is described that arises in a make-to-order steel company – a case study.


A new approach (called strip approach) is proposed in which the two-dimensional problem is considered as two one-dimensional problems.

### 5.3 Sequencing and assortment problems

Besides the waste minimization a suitable sequence of cutting the patterns may have remarkable effects in various practical situations. Furthermore, choosing appropriate sizes of the stock material may yield a waste reduction.


A two-stage procedure to solve the two-dimensional pattern-allocation problem is suggested. The first stage consists of solving the cutting stock problem without the sequencing constraint. In the second stage a sequencing problem (formulated as a traveling salesman problem) is used for the ordering of the cutting patterns in an optimal or near-optimal way.


In this two–dimensional assortment problem, a unit of product that is larger or better on two measurements or quantities may be used to substitute, at some cost, for a unit of demand that is smaller or inferior on both measurements or quantities. The problem is to determine which dimensional combinations to stock in order to minimize the combined stocking and substitution costs. Three heuristics are compared.


Various problems of a suitable sequencing of patterns are also investigated in the above papers.


In this article a comparison is made between three common heuristic methods for solution of the assortment problem.


A mixed-integer non-linear programming model is formulated to compute a large rectangle of minimum area containing a given set of small rectangles.


A heuristic to find the best sizes of stock sheets to minimize the trim loss is given.


The problem of determining the optimal length of a board with fixed width from which panels have to be cut to satisfy some specified demand with minimum waste is analyzed. The proposed procedure generates a lower bound on the best solution that can be obtained for each of the board length values within the allowed range.


The authors address the problem of determining what rectangular sizes should be stocked in order to satisfy a bill of materials composed of smaller rectangles.

6 General Two-dimensional C&P

In contrast to the so-called rectangular case where only rectangular items and objects are considered, now items with other regular or irregular shapes have to be arranged on not-necessary rectangular-shaped objects. The packing of items within a strip of minimal length is included as well as the packing of a maximum number of circles, etc.
6.1 Strip packing and layout problems


Some basic complexity results and solution approaches are summarized.


An approach for interactive defining of two-dimensional irregular shapes is presented.


These papers describe heuristic approaches for problems occurring in the clothing industry.

6.2 Non-rectangular C&P problems


The computation of optimal two-stage cutting patterns for trapezoids is considered.


For placements of various shapes, it is shown how to determine a cost function, annealing parameters and performance.


This paper describes the principal dimensional efficiency and cutting stock problems associated with the manufacture of communication and power cables. Some unique mathematical programming problems are identified and practical solution approaches to some of the new problems are presented.


The paper describes a real feasibility study of applying large-scale optimization methods to the cutting stock problem of irregular shapes.

Mixed-integer models are presented for two-dimensional packing problems of convex and non-convex polygons.


Heuristics to pack identical cylinders on a pallet are proposed.


Heuristics for the cylindrical bin packing problem are presented.


Optimal solutions and a method to prove their optimality are presented.


A survey is given of some of the literature concerning cutting and packing problems with irregular pieces.

7 Three-dimensional C&P

Three-dimensional C&P problems cover the so-called container loading and the multi-container loading problem. In general, a selection of various rectangular-shaped boxes have to be packed within a container maximizing the volumetric utilization or minimizing the number of containers required to pack all boxes.


Results of a simulation study into the stacking and handling of containers with the same dimensions are reported. The measures of performance include volumetric utilization, wasteful handling ratios, storage ratio, and rejection ratio.


A three-dimensional cutting problem is investigated - a case study.

A three–dimensional packing problem is investigated where one tries to pack various types of boxes (with different sizes) in various types of containers. The total cost of containers used is minimized.


The paper presents a complex computer-based heuristic procedure for sizing customer orders and developing three-dimensional load diagrams for rail and truck shipment.


The rectangular boxes are packed in a layer-wise manner.


Practical aspects are considered.


The application of methods to 3D problems is presented including that of a pallet chart.


Several on-line packing algorithms, called level-strip algorithms, are proposed and analyzed. Each of these algorithms has a separate heuristic in each dimension.


A heuristic based on the forward state strategy is proposed.


A heuristic and upper bounds are presented.


The modeling procedure presented converts the three-dimensional pallet loading problem into a standard mixed 0-1 integer programming model.


A heuristic has been developed to cope with the three-dimensional palletization problem in which boxes have different base dimensions and can be grouped by the same height
(layered-pallet-loading technique).


The heuristic presented is based on an integer programming formulation that uses the fractional knapsack problem as a column generation subproblem.


Some heuristics are presented together with some computational results.


The paper describes an efficient method of packing boxes into a container using a unique spatial representation technique.


The new heuristic aims to generate efficient loading arrangements which at the same time also provide a high degree of inherent stability.


Two approaches are proposed to obtain stable, evenly distributed packing patterns and to cater multi-drop loads.


The problem is formulated as a zero-one mixed integer programming model. It includes the consideration of multiple containers, multiple carton sizes, carton orientations, and the overlapping of cartons in a container.

8 Related Topics

In this section various references concerning special topics related to C&P are collected.


The problem of deciding whether a polygon $P$ can be translated to fit inside another polygon $P'$ is considered. Polynomial algorithms are presented for two cases: when both
$P$ and $P'$ are rectilinearly convex, and when $P$ is convex and $P'$ is arbitrary.


It is shown that, given a convex polygon $P$ with $n$ vertices, the densest lattice packing of $P$ in the plane can be found in $O(n)$ time.


Optimal algorithms are presented for solving the rectangle covering problem for monotone orthogonal polygons.


Theoretical investigations and heuristic approaches are presented for the multidimensional knapsack problem.


An adaptation of the greedy algorithm for the one, two and three dimensional cutting stock problem is proposed. The efficiency of the algorithms is measured in terms of the worst case bound and the time complexity.


The authors present a new and conceptually simple approach for finding exact solutions to large set covering, packing and partitioning problems.


Given two simple polygons, the algorithm finds the relative position of one with respect to the other such that the area of their convex enclosure is minimized.


Lower bounds on the asymptotic worst case ratio of any on-line d-dimensional vector packing algorithm are given.

A heuristic adapted from the first fit decreasing rule is proposed, and lower bounds for optimal solutions are investigated.


Finite and infinite packings of translates for centrally symmetric convex bodies in the d-dimensional Euclidean space are considered.


The book contains the mathematical background of regular parqueting as well as various applications in crystallography and industry.


For a special cutting problem the regions of stability of an optimal solution are obtained by a slight modification of a dynamic programming procedure used for solving the unperturbed problem.


The author proves that any set of similar triangles with total area equal to \( \alpha \) can be packed, using only translations and reflections, inside a similar triangle of area \( 2\alpha \). The bound \( 2\alpha \) cannot be improved in general.

### 9 Software

In this section various papers are contained concerning C&P software, concepts of program packages, comparisons of algorithms and numerical experience.


The paper documents empirical investigation and presents the results for 28 programs of different firms. (12 refs.)


Results are presented of a comparative analysis of fourteen heuristic packing rules.

Several commercial software packages for pallet loading are analyzed and compared.


The authors present two approaches to solve a two-dimensional cutting problem in CHIP and compare them with the standard ones. It turns out that, although CHIP greatly simplifies the problem statement, it is comparable in efficiency to specialized programs.


In this survey, three major trends of engineering activities are identified and discussed in detail: integration of applications, shift from optimization to control, and construction of new related problems. Moreover a learning expert system CUT is proposed which combines Simulated Annealing and Case Based Reasoning.


An interactive system, developed for a Portuguese steel manufacturer, is presented.


A container loading software package for a pulp and paper manufacturer is described.


Based on a branch-and-bound concept, boxes of various sizes are packed on a minimum number of pallets regarding several packing constraints of practical interest.