A Note on Handling Residual Lengths

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Summary: In this note the handling of utilizable residual lengths is investigated for the one-dimensional cutting stock problem. The corresponding continuous relaxation problem is solved using the column generation technique. Finally, extensions to the two-dimensional cutting stock problem are discussed.

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Introduction

DYCKHOFF proposed in [1] a new linear programming approach to the cutting stock problem. This approach is usually called the “one-cut” approach or the “one-cut” model. One of the advantages, as suggested in [1], is the possibility of handling of residual lengths. (During the cutting of stock lengths into desired pieces there may be obtained parts of waste which are large enough to be useful in the future. Such waste is called residual length.)

The aim of this note is to show how such residual lengths can be taken into consideration using the common linear programming model (denoted as the “multi-cut” model in [2], [3]).

First of all, we restrict our investigations on the one-dimensional case. Pieces $T_i$ of length $l_i$ and order demand $h_i$, $i = 1, \ldots, m$, are to cut from stock lengths of sizes $L_q$, $q = 1, \ldots, p$. Each stock length possesses a value denoted by $v_q$. Additionally, some residual lengths $R_k$ of sizes $r_k$ have a positive valuation $w_k$, $k = 1, \ldots, r$. Find a set of cutting patterns and corresponding quantities, how often the patterns are to be cut, such that the order demands are fulfilled and the total cost (the cost of used stock lengths minus the value of the obtained residual lengths) is minimal.

It is assumed that sufficiently many stock lengths are available. (Problems dealing with restrictions on the availability of stock lengths are considered in [7].)
Mathematical Model

A feasible cutting pattern of stock length $L_q$ can be uniquely represented by an $(m + r)$-dimensional non-negative integer vector $a = (a_1, \ldots, a_{m+r})^T$ with

$$\sum_{i=1}^{m} l_i a_i + \sum_{k=1}^{r} r_k a_{m+k} \leq L_q.$$ 

Let $a^{iq} = (a_1^{iq}, \ldots, a_{m+r}^{iq})^T$ denote the $j$-th cutting pattern of the $q$-th stock length, $j = 1, \ldots, n_q$, $q = 1, \ldots, p$, where $n_q$ is the number of admissible cutting patterns of length $L_q$. In the following model the integer variable $x_{jq}$ gives the number of times how often the cutting pattern $a^{iq}$ is cut.

$$z = \sum_{q=1}^{p} v_q \sum_{j=1}^{n_q} x_{jq} - \sum_{q=1}^{p} \sum_{j=1}^{n_q} \sum_{k=1}^{r} w_k a_{m+k}^{iq} x_{jq} = \min \quad (1)$$

subject to

$$\sum_{q=1}^{p} \sum_{j=1}^{n_q} a_{iq} x_{jq} \geq b_i, \quad i = 1, \ldots, m, \quad (2)$$

$$x_{jq} \geq 0, \quad j = 1, \ldots, n_q, \quad q = 1, \ldots, p, \quad (3)$$

$$x_{jq} \text{ integer}, \quad j = 1, \ldots, n_q, \quad q = 1, \ldots, p. \quad (4)$$

Hence, the mathematical model has the same structure as the usually used model. Therefore, one can apply the same solution strategy. Because of the exponential number of variables in problems of medium size the continuous relaxation model (relaxing (4)) is used to obtain nearly optimal integer solutions. Thereby the revised simplex method and column generation technique can be applied (see [3], [4], [8] or [9]).

Let be given a set of $m$ linear independent column vectors which forms a feasible basis $B$ of the relaxation problem (i.e. $B^{-1} b \geq 0$) and let $d = (d_1, \ldots, d_m)^T$ denote the vector of the simplex multipliers. Then the column generation problem is the following:

$$c^* = \min \left\{ \min_{q=1, \ldots, p} z_q, \min_{i=1, \ldots, m} d_i \right\} \quad (5)$$

where

$$z_q = \min \left\{ v_q - \sum_{k=1}^{r} w_k a_{m+k}^q - \sum_{i=1}^{m} d_i a_i^q \mid \sum_{i=1}^{m} l_i a_i^q + \sum_{k=1}^{r} r_k a_{m+k}^q \leq L_q, \quad a_i^q \geq 0, \text{ integer, } i = 1, \ldots, m+r \right\}, \quad (6)$$

$$q = 1, \ldots, p.$$ 

(The term $\min_{i=1, \ldots, m} d_i$ in (5) corresponds to the slack variables which are introduced in (2).)
If \( c^* \geq 0 \), then the current basis \( B \) (the set of chosen cutting patterns) is an optimal one with respect to the relaxation problem. Otherwise, a new column (a cutting pattern or a unit vector corresponding to a slack variable) is obtained and the simplex algorithm has to continue.

The Computation of \( z_q \)

Let

\[
f(l) = \max \left\{ \sum_{i=1}^{m} d_i a_i \left| \sum_{i=1}^{m} l_i a_i \leq 1, \ a_i \geq 0, \ \text{integer, } i = 1, \ldots, m \right. \right\}. \tag{7}
\]

\( f(l) \) is the optimal value of the knapsack problem with right hand side \( l \). Using (7) it is easy to verify that (6) is equivalent to the following formula:

\[
z_q = v_q - \max \left\{ \sum_{k=1}^{r} w_k a_{m+k} + f \left( L_q - \sum_{k=1}^{r} r_k a_{m+k} \right) \right\}
\]

\[
\sum_{k=1}^{r} r_k a_{m+k} \leq L_q,
\]

\[
a_{m+k} \geq 0, \ \text{integer, } k = 1, \ldots, r, \right\},
\]

\[q = 1, \ldots, p.\tag{8}\]

Formula (8) suggests some reasonable properties on the valuation of the residual lengths. Firstly, to avoid a "useless" cut up of stock lengths in residual lengths, it is assumed that

(i) \( w_k / r_k \leq v_q / L_q \) for all \( k \) and \( q \) with \( r_k \leq L_q \).

(ii) \( w_k / r_k \leq w_t / r_t \) for all \( k \) and \( t \) with \( r_k \leq r_t \).

(i) and (ii) seem to be natural conditions. Secondly, it is assumed that

(iii) if \( r_k \) and \( r_t \) are residual lengths then \( r_k + r_t \) is also a residual length (for all \( k \) and \( t \)).

With (iii) is guaranteed, in the case when residual lengths \( r_k \) and \( r_t \) may be obtained from a cutting pattern, that there exists another pattern, yielding residual length \( r_k + r_t \), which has a valuation not worse than the first one. Because of the assumptions (i), (ii) and (iii) there follows:

There exists an optimal solution of (8) with

\[
\sum_{k=1}^{r} a_{m+k} \leq 1. \tag{9}
\]

Hence, formula (8) can be reduced to

\[
z_q = v_q - \max \left\{ f(L_q), \ w_k + f(L_q - r_k) \mid r_k \leq L_q, \ k = 1, \ldots, r \right\},
\]

\[q = 1, \ldots, p.\tag{10}\]
If (9) is required a priori then (iii) is redundant.

Summarizing, if (i), (ii) and (iii) or (9) are valid then

\[
\begin{align*}
    c^* &= \min \left\{ \min_{i=1, \ldots, m} d_i, \right. \\
    &\left. \min_{q=1, \ldots, p} v_q - \max \left\{ f(L_q), w_k + f(L_q - r_k) \mid r_k \leq L_q, k=1, \ldots, r \right\} \right\}. 
\end{align*}
\]  

(11)

If \( f(l) \) is computed for all \( l \) with \( l \leq \max_{q=1, \ldots, p} L_q \) using dynamic programming or the “forward state strategy” (see e.g. in [6] or [9]) then the solution of (11) takes only an additional expense of \( O(pr) \) computational operations in comparison to the standard cutting stock problem where \( f(l) \) is also to compute for all \( l = L_q, q=1, \ldots, p \).

The following trivial example points out difficulties when handling residual lengths.

Example 1

Let \( p = 1, \ L_1 = 10, \ v_1 = 10, \ m = 1, \ l_1 = 4, \ b_1 = 3, \ r_1 = 1, \ r_1 = 6, \ w_1 = 4 \). The optimal solution of the relaxation problem (1)–(3) is:

\[ 1.5 \text{ times } a^1 = (2; 0)^T \text{ with } z^* = 15. \]

In comparison, the obvious solution

\[ 1 \text{ times } a^1 \text{ and } 1 \text{ times } a^2 = (1; 1)^T \]

has the value \( z = 16 \).

If \( w_1 \) is changed to \( w_1 = 6 \) then

\[ 3 \text{ times } a^2 \text{ with } z = 12 \]

is the optimal solution, but here 3 stock lengths are required.

Hence, greatest care is needed when choosing the valuations of the residual lengths. Moreover, to get an integer solution from the optimal solution of the relaxation problem a suitable rounding procedure is wanted. Such an algorithm without regard to residual length is given in [9]. An adaption of this rounding procedure has additionally to take into consideration the residual lengths. Especially, the number of patterns may increase. The corresponding rules should regard the structure properties of the quotients \( w_k/r_k \) (e.g. “convexity” or “concavity”).

In the next example a strategy for handling residual lengths is used which seems to be best suitable. (We assume (9).)

Example 2

Let \( p = 1, \ L_1 = 20, \ v_1 = 20, \) (stock length)
\[ m = 3, \ l_1 = 11, \ b_1 = 20. \]
\[ l_2 = 6, \quad b_2 = 30, \quad (\text{pieces}) \]
\[ l_3 = 4, \quad b_3 = 10, \]
\[ r = 2, \quad r_j = 5, \quad w_i = 4, \quad (\text{residual lengths}) \]
\[ r_2 = 8, \quad w_2 = 7. \]

At first the optimal solution of the continuous relaxation problem is computed without regard to the residual lengths.

In the example the "continuous" solution equals the integer solution:

20 times \( a^1 = (1, 1, 0; 0, 0)^T \),

(0 times \( a^2 = (0, 3, 0; 0, 0)^T \),

5 times \( a^3 = (0, 2, 2; 0, 0)^T \).

The optimal value is \( z^* = 500 \). 25 stock lengths are cut and 60 units of length are (useless) waste.

Starting with the basis

\[
B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1/3 & 1/3 & -1/3 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad d = \begin{pmatrix} 40/3 \\ 20/3 \end{pmatrix}
\]

one obtains with

\[
f(l) = \max \{ 10/3(4a_1 + 2a_2 + a_3); 11a_1 + 6a_2 + 4a_3 \leq l, \quad a_i \geq 0, \text{ integer, } i = 1, 2, 3 \};
\]

\[
f(20) = 20, \quad f(15) = 40/3, \quad f(12) = 40/3.
\]

Hence, \( c^* = \min \{ 20 - f(20), \quad 20 - 4 - f(15), \quad 20 - 7 - f(12) \} = -1/3 \). The corresponding column vector is \( a^4 = (1, 0, 0; 0, 0)^T \).

Continuing the simplex method the optimal solution with regard to the residual lengths gives:

10 times \( a^4 = (1, 0, 0; 0, 1)^T \),

15 times \( a^5 = (0, 2, 0; 0, 1)^T \),

10 times \( a^6 = (1, 0, 1; 1, 0)^T \).

Now, \( z^* = 485 \) and only 10 units of length are useless waste. On the other hand, 35 (instead of 25) stock lengths are cut.

**Extensions to Two-Dimensional Problems**

In the case of the general two-dimensional guillotine cutting problem the handling of residual areas can be done in complete analogy. Using the recurrence formulas given by Gilmore and Gomory [4], [5] the additional expense to solve the generation problem (in comparison to the case without handling residual areas) is not greater than the additional effort in the one-dimensional case. In detail, let \( L_q \) and \( W_q \) denote the lengths and widths of stock plates
(q = 1, ..., p) and r_k and s_k the length and width parameters of residual areas (k = 1, ..., r). If f(l, w) represents the optimal value cutting of a rectangle of length l and width w in given rectangular pieces using guillotine cuts, then the formula equivalent to (10) is the following:

\[
\begin{align*}
    z_q &= v_q - \max \{ f(L_q, W_q), w_k + f(L_q - r_k, s_k) + f(L_q, W_q - s_q), \\
    &\quad w_k + f(L_q - r_k, W_q) + f(r_k, W_q - s_k), \\
    &\quad r_k \leq L_q, s_k \leq W_q, k = 1, \ldots, r \}, \\
    q &= 1, \ldots, p.
\end{align*}
\]

In the case of two- or three-stage guillotine cutting residual areas can easily be handled in a similar way if the residual areas are obtained by only one guillotine cut parallel to the cutting direction of the first stage. (The computation of an optimal cutting pattern for the 2- or 3-stage problem consists in the solution of several knapsack problems. Hence, a modification of (10) is applicable.) The handling of more general residual areas yields to a greater expense for solving the generation problems but is, is principle, possible.

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References


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