Multiplicity results for free and constrained nonlinearly elastic rods based on nonsmooth critical point theory

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Introduction

In many applications in the calculus of variation the existence of critical points, distinct from minima, which solve some basic equations (e.g., equilibrium equations) are of great interest. At the beginning of our century the papers by Fischer [26], Weyl [46], Courant [23], Morse [34], Ljusternik [29], and Schnirelman [38] were the birth of a new fascinating theory, the so called Critical Point Theory or Calculus of Variation in the Large, investigating such questions. Here in a very fruitful way topological arguments are exploited in analytical constructions. Undergoing a tremendous development in the second half of this century by contributions of Krasnoselskii, Vainberg, Schwartz, Palais, Smale, Browder, Ambrosetti, Rabinowitz, Conley, Zeidler, Struwe, Benci and many others, today we have an excellent theory representing some highlight in the calculus of variation (a survey can be found in [31]).

In particular in the Minimax Theory, for the determination of critical points of a functional \( f : X \to \mathbb{R} \) we consider the minimax value

\[
c := \inf_{F \in \mathcal{F}} \sup_{u \in F} f(u)
\]

where \( \mathcal{F} \) is some suitable class of subsets of \( X \). Then the existence of a critical point can be ensured, roughly speaking, if

(a) there exist special deformations of \( X \) such that \( f \) decreases substantially along these deformations and

(b) the class \( \mathcal{F} \) expresses some topological structure of the level sets of \( f \) and is invariant under the deformations according to (a).

The first condition is a pure technical one and can be satisfied if \( f \) has some smoothness and fulfills the so-called Palais-Smale condition, which expresses certain compactness. Then the number of critical points, which can be verified this way, essentially depends on the number of topologically different classes \( \mathcal{F} \) satisfying (b).

During the last 30 years there was a great deal of work to extend this minimax principle to more general spaces \( X \) and, in particular, to weaken the smoothness assumptions needed for condition (a). The main idea was to construct a so-called pseudo-gradient field associated with \( f \) and to integrate it over \( X \), i.e., the corresponding flow then gives the desired deformation. This way condition (a) can be verified for \( C^1 \)-functionals \( f \) on \( C^{1,1} \)-Finsler manifolds (cf. Ambrosetti [1]). Today it is a common opinion that the minimax principle is restricted to situations of this kind and that it is not applicable, e.g., to the interesting problems in modern nonlinear elasticity where the energy functional is naturally nonsmooth in general. Though there are some results extending the idea of pseudo-gradient fields to more general cases (Szulkin [45] - sum of convex and \( C^1 \)-functionals, K.C. Chang [17] and Schuricht [39] - locally Lipschitz continuous functionals, Miersemann [32], Schuricht [40], and Struwe [44] - deformations within convex sets), they are still too restrictive for many applications. In particular, a rigorous extension to general unilateral conditions occurring, e.g., in obstacle problems seems hardly possible. Thus it appears that the end of the road provided by this method is reached.

The described situation, however, is not satisfactory. On the one hand any smoothness restrictions seem to be artificial, because the minimax principle is essentially based on pure topological
arguments. Roughly speaking, to verify a mountain pass as in the famous theorem of Ambrosetti and Rabinowitz, which is as simple as brilliant, no differentiability should be necessary. On the other hand it is not completely clear what the critical points of a nonsmooth functional should be. A completely different approach to these problems was developed by Corvellec, Degiovanni and Marzocchi [24, 22] and, independently, by Ioffe, Katriel and Schwartzman [27, 28]. For merely continuous functionals \( f \) on a metric space \( X \) the weak slope \( |df|(u) \), replacing \( \|f'(u)\| \) in the smooth case, is introduced by means of local deformations of \( X \) near \( u \). On the basis of this notion critical points are defined and the existence of deformations as in (a) can be verified without the technicalities of pseudo-gradient fields. This construction even can be generalized to some classes of lower semicontinuous functionals. This way the main results of the minimax theory can be extended to a very general setting.

Nonsmooth methods are not known too much and many people feel uneasy in working with this subject. Moreover they suspect that the abstract constructions are not really applicable and they fear that the generalized critical points correspond to irregular solutions without practical relevance. The main goal of this paper is to demonstrate that, for nontrivial buckling problems in nonlinear elasticity, we do not have to worry about this point. In particular we study the buckling of shearable nonlinearly elastic rods, also under the presence of rigid obstacles. We verify the abstract assumptions of the nonsmooth theory and finally show that the obtained critical points satisfy the Euler-Lagrange equations, which are equivalent to the physical equilibrium conditions. This way we get multiplicity results combined with the justification that the solutions are reasonable. The application of the abstract theory involves, however, a great deal of technicality. Let us still mention that we do not restrict our attention to the study of variational inequalities in the case of unilateral obstacles. Based on a very general rod theory we employ a different approach developed in Schuricht [42]. By Clarke’s calculus of generalized gradients we determine the detailed structure of the Lagrange multiplier corresponding to the obstacle and obtain the Euler-Lagrange equations also in that case. Using a general concept for forces they coincide with the equilibrium conditions of the rod theory and allow a consistent and very natural physical interpretation of the contact forces.

Except for some generalization concerning the nonsmoothness of the stored energy function, the obtained buckling results for the case without obstacles are essentially known by the fascinating global bifurcation analysis of Antman & Rosenfeld [11], who used ideas of the global bifurcation alternative of Rabinowitz and the Sturm-Liouville theory. The results for the obstacle problem, however, are completely new and demonstrate the necessity as well as the applicability of nonsmooth methods very nicely.

In our applications we work with the geometrically exact Cosserat theory for the deformation of nonlinearly elastic rods (cf. Antman [7]). This is a mathematically one-dimensional theory which can be derived exactly from three dimensional elasticity. It describes rods that can suffer flexure, extension, and shear and it involves a general nonlinear constitutive relation. As we shall see this theory has a much richer structure than the mostly used Euler elastica (or simplifications of it). For our purposes it is sufficient to study planar deformations, though lateral buckling may occur (cf. Antman [6], Maddocks [30]). We show, roughly speaking, that for an axially compressed originally straight rod, possibly subjected to a symmetric rigid obstacle, the number
of nontrivial buckling states increases under an increasing compressing force when the material satisfies certain stiffness condition. Otherwise it can happen that, e.g., for a very soft material, the rod becomes shorter and shorter and does never buckle (cf. Antman & Pierce [10]). In contrast to standard applications of Ljusternik-Schnirelman theory, where the existence of infinitely many solutions on some level set is verified (which is useful for bifurcation assertions), our multiplicity results seem to have more practical relevance, in particular in the light of the rich structure of the global bifurcation behavior, comprehensively investigated by Antman & Rosenfeld [11] for the case without obstacles. Instead of proving bifurcation results we rather claim to demonstrate how to handle nonsmooth constitutive rules and general rigid obstacles by our variational methods, and we extend all previous results this way.

In addition to the previously mentioned papers concerning buckling of Cosserat rods, we refer also to Antman [4], Antman & Marlow [8], [9]. Buckling with respect to unilateral constraints was studied for more primitive rod models by, e.g., Degiovanni, Kučera, Link, Miersemann, Mittelmann, Quittner, Schuricht, and Zeidler (for references see Miersemann [33], Schuricht [42]). General discussions about buckling problems, including historical remarks and extensive references, can be found in Antman [3], [5].

In the first section of this paper the notion of weak slope and some basic properties are introduced. The second section mainly provides an adaption of a classical critical point theorem of D.C. Clark [18] for the symmetric case to the equivariant situation of our applications (see also Rabinowitz [35]). The advantage of this result of Clark is that the construction of certain small variations of some trivial solution is sufficient for the existence of “large” buckled states. The technical assumptions as the Palais-Smale condition and an epigraph condition are studied in Section 3 for a more specific setting, as met in the applications, and sufficient conditions, mainly based on transversality and compactness, are derived.

The Cosserat theory for the planar deformation of rods is introduced in Section 4 in an extended version as needed for obstacle problems where, in particular, more general forces than usually can occur (cf. Schuricht [42]). Moreover we discuss how to handle certain nonsmooth constitutive relations expressing some kind of plasticity. In Section 5 we apply our abstract results to the buckling of an axially compressed originally straight rod which is fixed at the left end and can slide along the horizontal axis with the right end, but without obstacles. Always we have two trivial solutions which corresponds to a straight and compressed or stretched configuration. Under certain material restrictions we then obtain any number of different nontrivial buckled states as soon as the load is large enough. The verification of the Euler-Lagrange equations finally shows that our solutions are reasonable. Section 6 is devoted to the same problem but subjected to obstacles. In the case of symmetric obstacles we get analogous multiplicity results as in the free case. For the nonsymmetric case, which needs a more detailed study of the specific problem, we provide a general minimax result which can be adapted to special situations. Following the line in Schuricht [42], again the Euler-Lagrange equations are derived for all determined solutions.

To verify the assumptions of our abstract theory in Sections 5 and 6, very subtle technical constructions using variations only on suitable measurable subsets are necessary, since we allow nonhomogenous and nonsmooth constitutive relations. As in Schuricht [42] we again observe that a certain intrinsic formulation of the rod problem is very convenient, while the extrinsic
formulation is probably not applicable. This roughly means that we have to formulate the problem in $L^p$ spaces and not, as usually, in Sobolev spaces $W^{1,p}$. By this reason we however get long and nasty formulas for simple facts and the proofs look terribly technical. Therefore we first present the case without obstacles and afterwards the constrained case. This way the reader can easier overview the analysis and recognize the additional effort needed in the presence of obstacles.

Let us finally emphasize that one of our main goals in this paper is to demonstrate the applicability of modern nonsmooth critical point theory in nonlinear elasticity.

1 Tools of nonsmooth analysis

In this section we provide some tools from nonsmooth analysis as basis for our nonsmooth critical point theory. Let $X$ be a metric space endowed with the metric $d$. In the following, $B_r(u)$ will denote the open ball of centre $u$ and radius $r$.

The next notion has been independently introduced in [22, 24] and [28], while a variant can be found in [27].

**Definition 1.1** Let $f : X \to \mathbb{R}$ be a continuous function. For every $u \in X$ we denote by $|df|(u)$ the supremum of the $\sigma$’s in $[0, +\infty]$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(u) \times [0, \delta] \to X$ such that

\[
\forall v \in B_\delta(u), \forall t \in [0, \delta] : d(\mathcal{H}(v, t), v) \leq t, \tag{1.2}
\]

\[
\forall v \in B_\delta(u), \forall t \in [0, \delta] : f(\mathcal{H}(v, t)) \leq f(v) - \sigma t. \tag{1.3}
\]

The extended real number $|df|(u)$ is called the weak slope of $f$ at $u$.

It is easy to see that the function $|df| : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

Now consider a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$. Set

\[
\forall b \in \mathbb{R} : f^b = \{u \in X : f(u) \leq b\},
\]

\[
\mathcal{D}(f) = \{u \in X : f(u) < +\infty\},
\]

\[
\text{epi}(f) = \{(u, \xi) \in X \times \mathbb{R} : f(u) \leq \xi\}
\]

and define a function $\mathcal{G}_f : \text{epi}(f) \to \mathbb{R}$ by $\mathcal{G}_f(u, \xi) = \xi$. The set $X \times \mathbb{R}$ will be endowed with the metric

\[
d((u, \xi), (v, \mu)) = (d(u, v)^2 + (\xi - \mu)^2)^{1/2}
\]

and $\text{epi}(f)$ with the induced metric. According to [22, 24], let us give the following

**Definition 1.4** For every $u \in \mathcal{D}(f)$ let

\[
|df|(u) = \begin{cases} 
\frac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}} & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1 \\
+\infty & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1
\end{cases}
\]
When \( f \) is real valued and continuous, the above definition turns out to be consistent with Definition 1.1 (see [24]). Let us recall from [15, Theorem 1.5.4] a criterion to obtain an estimate of \(|dG_f|(u, \xi)\) and \(|df|(u)\) in the lower semicontinuous case.

**Proposition 1.5** Let \((u, \xi) \in \text{epi}(f)\) and let \(\sigma \geq 0\). Assume there exist \(\delta > 0\) and a continuous map

\[
\mathcal{H} : \{ v \in B_\delta(u) : f(v) < \xi + \delta \} \times [0, \delta] \to X
\]
such that

\[
\forall v \in B_\delta(u), \forall t \in [0, \delta] : f(v) < \xi + \delta \implies d(\mathcal{H}(v, t), v) \leq t,
\]

\[
\forall v \in B_\delta(u), \forall t \in [0, \delta] : f(v) < \xi + \delta \implies f(\mathcal{H}(v, t)) \leq f(v) - \sigma t.
\]

Then we have \(|dG_f|(u, \xi) \geq \frac{\sigma}{\sqrt{1 + \sigma^2}}\). In particular, if \(\xi = f(u)\), it is \(|df|(u) \geq \sigma\).

In the following, we shall need also a result concerning the sum of two functions. It is an extension of [21, Theorem 2].

**Proposition 1.6** Let \(f_0 : X \to \mathbb{R} \cup \{+\infty\}\) be a lower semicontinuous function, \(f_1 : X \to \mathbb{R}\) a locally Lipschitz continuous function, \(f = f_0 + f_1\) and \(u \in D(f_0)\). Let \(L\) be a Lipschitz constant for \(f_1\) in a neighbourhood of \(u\).

Then for every \(\xi \in \mathbb{R}\) with \((u, \xi) \in \text{epi}(f_0)\) we have

\[
|dG_{f_0}|(u, \xi) < 1 \implies \left| \frac{|dG_{f_0}|(u, \xi + f_1(u))}{\sqrt{1 - |dG_{f_0}|(u, \xi + f_1(u))^2}} - \frac{|dG_{f_0}|(u, \xi)}{\sqrt{1 - |dG_{f_0}|(u, \xi)^2}} \right| \leq L.
\]

In particular, it is

\[
|df|(u) < +\infty \iff |df_0|(u) < +\infty,
\]

\[
|df_0|(u) < +\infty \implies \|df|(u) - |df_0|(u)\| \leq L.
\]

**Proof.** Let \(0 < \sigma < |dG_{f_0}|(u, \xi)\) and let

\[
\mathcal{H} : (\text{epi}(f_0) \cap B_\delta(u, \xi)) \times [0, \delta] \to \text{epi}(f_0)
\]

be a continuous map as in Definition 1.1 where we use the decomposition \(\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2)\) according to \(X \times \mathbb{R}\). Without loss of generality, we can suppose that \(f_1\) is Lipschitz continuous of constant \(L\) on \(B_\delta(u)\). From the conditions

\[
d(\mathcal{H}_1((v, \mu), t), v)^2 + (\mathcal{H}_2((v, \mu), t) - \mu)^2 \leq t^2,
\]

\[
\mathcal{H}_2((v, \mu), t) \leq \mu - \sigma t
\]

we deduce that

\[
d(\mathcal{H}_1((v, \mu), t), v) \leq \sqrt{1 - \sigma^2 t}.
\]

Let

\[
\vartheta = \sqrt{1 - 2\sigma L\sqrt{1 - \sigma^2} + L^2(1 - \sigma^2)}
\]
and let $\delta' > 0$ be such that $\delta' \leq \delta$ and such that $d((v, \mu - f_1(v)), (u, \xi)) < \delta$ whenever $d((v, \mu), (u, \xi + f_1(u))) < \delta'$. Define a continuous map

$$K : (\text{epi}(f) \cap B_{\delta'} (u, \xi + f_1(u))) \times [0, \delta'] \to \text{epi}(f)$$

by

$$K((v, \mu), t) = \left( \mathcal{H}_1 \left( (v, \mu - f_1(v), \frac{t}{\vartheta}) \right), \mu - \sigma \frac{t}{\vartheta} + L \sqrt{1 - \sigma^2 \frac{t}{\vartheta}} \right).$$

Since

$$f_0 \left( \mathcal{H}_1 \left( (v, \mu - f_1(v), \frac{t}{\vartheta}) \right) \right) + f_1 \left( \mathcal{H}_1 \left( (v, \mu - f_1(v), \frac{t}{\vartheta}) \right) \right) \leq \mathcal{H}_2 \left( (v, \mu - f_1(v), \frac{t}{\vartheta}) \right) + f_1 \left( \mathcal{H}_1 \left( (v, \mu - f_1(v), \frac{t}{\vartheta}) \right) \right) \leq \mu - f_1(v) - \sigma \frac{t}{\vartheta} + f_1 \left( \mathcal{H}_1 \left( (v, \mu - f_1(v), \frac{t}{\vartheta}) \right) \right) \leq \mu - \sigma \frac{t}{\vartheta} + L \sqrt{1 - \sigma^2 \frac{t}{\vartheta}},$$

the map $K$ takes actually its values in epi $(f)$. Moreover, we have

$$d(K((v, \mu), t), (v, \mu))^2 = d \left( \mathcal{H}_1 \left( (v, \mu - f_1(v), \frac{t}{\vartheta}) \right), v \right)^2 + \left( -\sigma \frac{t}{\vartheta} + L \sqrt{1 - \sigma^2 \frac{t}{\vartheta}} \right)^2 \leq \left( 1 - \sigma^2 \right) \frac{t^2}{\vartheta^2} + \left( -\sigma \frac{t}{\vartheta} + L \sqrt{1 - \sigma^2 \frac{t}{\vartheta}} \right)^2 = \frac{1 - 2\sigma L \sqrt{1 - \sigma^2} + L^2 (1 - \sigma^2)}{\vartheta^2} t^2 = \frac{1 - 2\sigma L \sqrt{1 - \sigma^2} + L^2 (1 - \sigma^2)}{\vartheta^2} t^2.$$ Finally, we also have

$$\mathcal{G}_f(K((v, \mu), t)) = \mu - \sigma \frac{t}{\vartheta} + L \sqrt{1 - \sigma^2 \frac{t}{\vartheta}} = \mathcal{G}_f(v, \mu) - \sigma - L \sqrt{1 - \sigma^2} \frac{t}{\vartheta}.$$

It follows

$$|d\mathcal{G}_f|(u, \xi + f_1(u)) \geq \frac{\sigma - L \sqrt{1 - \sigma^2}}{\sqrt{1 - 2\sigma L \sqrt{1 - \sigma^2} + L^2 (1 - \sigma^2)}}. \quad (1.7)$$

Now, if $|d\mathcal{G}_{f_0}|(u, \xi) = 1$, we deduce that $|d\mathcal{G}_f|(u, \xi + f_1(u)) = 1$. Since $f_0 = f + (-f_1)$, the converse is also true. If $|d\mathcal{G}_{f_0}|(u, \xi) < 1$, then (1.7) is equivalent to

$$\frac{|d\mathcal{G}_f|(u, \xi + f_1(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, \xi + f_1(u))^2}} \geq \frac{\sigma}{\sqrt{1 - \sigma^2}} - L,$$

which implies

$$\frac{|d\mathcal{G}_f|(u, \xi + f_1(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, \xi + f_1(u))^2}} \geq \frac{|d\mathcal{G}_{f_0}|(u, \xi)}{\sqrt{1 - |d\mathcal{G}_{f_0}|(u, \xi)^2}} - L.$$ Again, by exchanging $f_0$ with $f$, we get

$$\left| \frac{|d\mathcal{G}_f|(u, \xi + f_1(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, \xi + f_1(u))^2}} - \frac{|d\mathcal{G}_{f_0}|(u, \xi)}{\sqrt{1 - |d\mathcal{G}_{f_0}|(u, \xi)^2}} \right| \leq L$$

and the proof is complete. \[\square\]
2 Nonsmooth critical point theory

In this section we modify and extend some abstract results of nonsmooth critical point theory to the case of lower semicontinuous and locally Lipschitz continuous functionals on metric spaces $X$ sufficient for our applications below.

**Definition 2.1** We say that $u \in D(f)$ is a (lower) critical point of $f$, if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a (lower) critical value of $f$, if there exists a (lower) critical point $u \in D(f)$ of $f$ with $f(u) = c$.

The idea of Definition 1.4 is to reduce the study of the lower semicontinuous function $f$ to that of the (Lipschitz) continuous function $G_f$. In view of critical point theory, it is suitable to introduce a condition which implies a bijective correspondence between the critical points of $f$ and those of $G_f$.

**Definition 2.2** We say that $f$ satisfies the condition $(\text{epi})$ on $Z \subset X$, if

$$\forall b > 0 : \inf \{|dG_f|(u, \xi) : u \in D(f) \cap Z, f(u) < \xi, |\xi| \leq b\} > 0.$$  

If $f$ is real valued and continuous, we have $|dG_f|(u, \xi) = 1$ whenever $f(u) < \xi$. The same property holds for some important classes of lower semicontinuous functions (see [16, 21, 22, 24]). Consequently, in such cases condition $(\text{epi})$ is clearly satisfied.

**Definition 2.3** Let $c \in \mathbb{R}$. A sequence $(u_n)$ in $D(f)$ is said to be a Palais-Smale sequence at level $c$ ($(PS)_c$-sequence, for short) for $f$, if $f(u_n) \to c$ and $|df|(u_n) \to 0$.

We say that $f$ satisfies the Palais-Smale condition at level $c$ ($(PS)_c$ for short) on $Z \subset X$, if every $(PS)_c$-sequence $(u_n)$ for $f$ with $u_n \in Z$ admits a convergent subsequence $(u_{n_k})$ in $X$.

We say that $f$ satisfies the Palais-Smale condition at level $c$, if it satisfies the Palais-Smale condition at level $c$ on $X$.

**Remark 2.4** Assume that $f$ satisfies condition $(\text{epi})$ and take $c \in \mathbb{R}$. Then $f$ satisfies $(PS)_c$ if and only if $G_f$ satisfies $(PS)_c$.

As usual, let us set for every $c \in \mathbb{R}$

$$K_c = \{u \in D(f) : |df|(u) = 0, f(u) = c\}.$$  

We now provide some critical point results we will apply below to buckling problems. In the presence of some symmetry, we use an adaptation of a classical result of D. C. Clark (see e.g. [18, 35]) to our setting.

Let $\Phi : X \to X$ be an isometry such that $\Phi^2 = \text{Id}$. We say that $f : X \to \mathbb{R} \cup \{+\infty\}$ is $\Phi$-invariant, if $f(\Phi(u)) = f(u)$ for any $u \in X$. If $S \subset \mathbb{R}^n$ is symmetric with respect to the origin, we say that $\psi : S \to X$ is $\Phi$-equivariant, if $\psi(-x) = \Phi(\psi(x))$ for any $x \in S$. Finally, we set

$$\text{Fix}(X) = \{u \in X : \Phi(u) = u\}.$$  

**Theorem 2.5** Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous $\Phi$-invariant function satisfying condition $(\text{epi})$ and let $k \in \mathbb{N}$. Assume that
(a) $f$ is bounded from below;

(b) there exists a continuous $\Phi$–equivariant map $\psi$ from the $(k-1)$–dimensional sphere $S^{k-1}$ to $X$ such that
\[
\sup \left\{ f(\psi(x)) : x \in S^{k-1} \right\} < \inf \left\{ f(u) : u \in \text{Fix}(X) \right\}
\]
(we agree that $\inf \emptyset = +\infty$);

(c) $X$ is complete and for any $c \in \mathbb{R}$ with $c < \inf \left\{ f(u) : u \in \text{Fix}(X) \right\}$ the function $f$ satisfies condition $(PS)_c$.

Then there exist at least $k$ distinct pairs $(u_1, \Phi(u_1)), \ldots, (u_k, \Phi(u_k))$ of critical points of $f$ with $f(u_j) < \inf \left\{ f(u) : u \in \text{Fix}(X) \right\}$.

The proof of this theorem is given at the end of this section.

In the case without symmetry, we can recall a result of mountain pass type (see [2, 35] for the smooth case).

**Theorem 2.6** Let $D^k$ and $S^{k-1}$ denote respectively the closed unit ball and sphere in $\mathbb{R}^k$, let $\psi : S^{k-1} \to D(f)$ be continuous and set
\[
\Gamma := \left\{ \gamma : D^k \to D(f) : \gamma \text{ is continuous and } \gamma = \psi \text{ on } S^{k-1} \right\}.
\]
Assume that
\[
\Gamma \neq \emptyset \quad \text{and} \quad \sup_{x \in S^{k-1}} f(\psi(x)) < c < +\infty,
\]
where
\[
c := \inf_{\gamma \in \Gamma} \sup_{x \in D^k} f(\gamma(x)).
\]

If $X$ is complete and $f$ satisfies conditions $(PS)_c$ and $(epi)$, then $c$ is a critical value of $f$.

**Proof.** It is a particular case of [22, Theorem (4.5)]. $\blacksquare$

For the application of the previous critical point theorems, we are interested in sufficient conditions for $(PS)_c$ and $(epi)$. In the next section we shall prove such sufficient conditions in a Banach space $Y$, instead of the metric space $X$ itself. The following proposition justifies that this is sufficient for a certain class of problems.

**Proposition 2.9** Let $Y$ be another metric space and $p : Y \to X$ a map. Assume that for every $v \in Y$ there exists a neighbourhood $V$ of $v$ such that $p(V)$ is a neighbourhood of $p(v)$ and $p : V \to p(V)$ is an isometry.

Then the following facts hold:

(a) if $(f \circ p)$ satisfies condition $(PS)_c$ on $Z \subset Y$ for some $c \in \mathbb{R}$, then $f$ satisfies condition $(PS)_c$ on $p(Z)$;

(b) if $(f \circ p)$ satisfies condition $(epi)$ on $Z \subset Y$, then $f$ satisfies condition $(epi)$ on $p(Z)$. 

PROOF. It is easy to verify that \( |dG_{f(p)}|(v,\mu) = |dG_f|(p(v),\mu) \) and \( |d(f \circ p)|(v) = |df|(p(v)) \) for each \((v,\mu) \in \text{epi}(f \circ p)\). Then both assertions readily follow. 

PROOF of Theorem 2.5. Suppose first that \( f \) is real valued and continuous. It is easy to see that \( |df|(\Phi(u)) = |df|(u) \) for any \( u \in X \). Moreover, if \( u \in X \setminus \text{Fix}(X) \) and \( 0 < \sigma < |df|(u) \), we can find \( \delta > 0 \) and a continuous map \( \mathcal{H} : B_\delta(u) \times [0,\delta] \to X \) as in Definition 1.1. Without loss of generality, we can assume that \( B_\delta(u) \cap B_\delta(\Phi(u)) = \emptyset \). Then the map

\[
\tilde{\mathcal{H}} : (B_\delta(u) \cup B_\delta(\Phi(u))) \times [0,\delta] \to X
\]

defined by

\[
\tilde{\mathcal{H}}(v, t) = \begin{cases} 
\mathcal{H}(v, t) & \text{if } v \in B_\delta(u) \\
\Phi(\mathcal{H}(\Phi(v), t)) & \text{if } v \in B_\delta(\Phi(u))
\end{cases}
\]

is continuous, satisfies (1.2), (1.3) and the further condition

\[
\forall v \in B_\delta(u) \cup B_\delta(\Phi(u)), \forall t \in [0,\delta] : \tilde{\mathcal{H}}(\Phi(v), t) = \Phi(\tilde{\mathcal{H}}(v, t)).
\]

Consequently, the same argument of [22, Theorem (2.16)] shows that for every \( c < \inf\{f(u) : u \in \text{Fix}(X)\} \), every \( \sigma > 0 \) and every neighbourhood \( \mathcal{O} \) of \( K_c \) there exist \( \varepsilon \in ]0,\sigma[ \) and a continuous map \( \eta : X \times [0,1] \to X \) such that

\[
d(\eta(u,t),u) \leq t, \\
f(\eta(u,t)) \leq f(u), \\
f(u) \notin [c-\varepsilon, c+\varepsilon] \implies \eta(u,t) = u, \\
\eta(f^{c+\varepsilon} \setminus \mathcal{O}, 1) \subset f^{c-\varepsilon}, \\
\eta(\Phi(u), t) = \Phi(\eta(u,t)).
\]

Now, the same argument of [35, Theorem 9.1] can be easily adapted to our setting (the notion of genus has also to be adapted in the obvious way) and the assertion follows.

In the general case, consider the complete metric space \( \text{epi}(f) \) and the continuous function \( G_f \). If we set

\[
\tilde{\Phi}(u, \xi) = (\Phi(u), \xi), \\
\alpha = \sup \left\{ f(\psi(x)) : x \in S^{k-1} \right\}, \\
\hat{\psi}(x) = (\psi(x), \alpha),
\]

it is easy to check that \( \tilde{\Phi} : \text{epi}(f) \to \text{epi}(f) \) is an isometry with \( \tilde{\Phi}^2 = \text{Id} \), \( G_f \) is \( \tilde{\Phi} \)-invariant and

\[
\inf \{ G_f(u, \xi) : (u, \xi) \in \text{Fix}(\text{epi}(f)) \} = \inf \{ f(u) : u \in \text{Fix}(X) \}.
\]

Moreover, all the other assumptions of the theorem are satisfied by \( G_f \), so that, by the previous step, we deduce the existence of \( k \) distinct pairs \(((u_1, \xi_1), (\Phi(u_1), \xi_1)), \ldots, ((u_k, \xi_k), (\Phi(u_k), \xi_k))\) of critical points of \( G_f \) with

\[
\xi_j < \inf \{ f(u) : u \in \text{Fix}(X) \}.
\]

By condition (epi), we have \( \xi_j = f(u_j) \) and the assertion easily follows. 

3 Constrained problems

We are interested in critical points of a functional \( f : X \to \mathbb{R} \cup \{+\infty\} \) on a metric space \( X \) subjected to constraints \( g_0(u) \leq 0 \) and \( g_1(u) = 0 \) where \( g_0, g_1 \) are real functionals on \( X \). The case without inequality constraint \( g_0(u) \leq 0 \), which we also consider in our applications, can be simulated by setting \( g_0(u) := -1 \) on \( X \).

In order to apply Theorem 2.5 and Theorem 2.6 to such problems, we claim to derive abstract conditions sufficient for \((PS)_c\) and \((epi)\) in this section. For this purpose, a local linear structure in the space \( X \) may help. Hence Finsler manifolds would be a suitable mathematical tool for our analysis. However, in our applications the metric space \( X \) is locally isometric with a piece of a fixed Banach space \( Y \) and, based on Proposition 2.9, it is sufficient to study conditions \((PS)_c\) and \((epi)\) on the Banach space \( Y \), instead of introducing the technicalities of Finsler manifolds.

As we shall see, transversality and compactness play an important role for \((PS)_c\) and \((epi)\) and we have to estimate \(|df|\) by means of other notions of nonsmooth analysis.

**Note.** Our buckling problems below are formulated in a metric space \( X \) that is the cross product of a Banach space and the sphere \( S^1 \), where the component in \( S^1 \) stands for some angle describing the direction of the left cross-section of the rod. Formally, we could replace \( S^1 \) by \( \mathbb{R} \) and would get a Banach space \( Y \). Then, however, we have an artificial multiplicity which prevents coercivity and, consequently, the Palais-Smale condition cannot be verified. But it is easy to see how Proposition 2.9 reduces the problem to the Banach space \( Y \).

Let now \( Y \) be a real Banach space, let \( f_0 : Y \to \mathbb{R} \cup \{+\infty\} \) be a lower semicontinuous convex function, let \( f_1, g_0, g_1 : Y \to \mathbb{R} \) be locally Lipschitz continuous functions, let \( f := f_0 + f_1 \), and set

\[
K := \{ u \in Y : g_0(u) \leq 0, g_1(u) = 0 \}.
\]

In the following, \( \partial f(u) \) will denote either the convex subdifferential or Clarke’s generalized gradient, \( f^0(u;v) \) the associated generalized directional derivative (see [19]) and \( I_F \) the indicator of a subset \( F \) of \( Y \).

Before studying the conditions \((PS)_c\) and \((epi)\), let us provide the following result, which enables us to verify that a certain trivial unbuckled state is always a critical point (subjected to different constraints and loads).

**Proposition 3.1** If \( f_1 \) is of class \( C^1 \), then the following facts hold:

(a) for every \( u \in D(f) \) we have that

\[
|df|(u) = \begin{cases} 
\min \{ \|f_0^* + f_1^*(u)\| : f_0^* \in \partial f_0(u) \} & \text{if } \partial f_0(u) \neq \emptyset, \\
+\infty & \text{if } \partial f_0(u) = \emptyset;
\end{cases}
\]

(b) if \( C \) is a closed subset of \( Y \), for every \( u \in C \cap D(f) \) we have that

\[
|df|(u) = 0 \implies |d(f + I_C)|(u) = 0.
\]

**Proof.**

(a) See [24, Proposition 2.10 and Theorem 2.11].
Let us now formulate a sufficient condition for (epi) also fundamental for the investigation of the Palais-Smale condition in the next subsection. For

\[ \text{The proof of this theorem is given below. It is based mainly on the following theorem, which is} \]

\[ \text{Theorem 3.4} \]

\[ \text{Let} \] \( C \)

\[ \text{be a convex subset of} \] \( Y \)

\[ \text{and let} \] \( u \in C \cap K \).

\[ \text{We say that} \] \( (C, g_0, g_1) \)

\[ \text{is transversal at} \] \( u \), \( \text{if there exist} \) \( u_-, u_+ \in C \) such that

\[ \begin{align*}
(a) \ & \text{if} \ g_0(u) < 0, \ \text{we have} \\
& g_0^0(u; u_- - u) < 0, \quad g_1^0(u; u - u_+) < 0; \\
(b) \ & \text{if} \ g_0(u) = 0, \ \text{we have} \\
& g_0^0(u; u_- - u) < 0, \quad g_1^0(u; u_- - u) < 0, \\
& g_0^0(u; u_+ - u) < 0, \quad g_1^0(u; u - u_+) < 0.
\end{align*} \]

\[ \text{We say that} \] \( (C, g_1) \)

\[ \text{is transversal at} \] \( u \), \( \text{if} \) \( (C, g_0, g_1) \) \text{is transversal at} \( u \) \text{for} \( g_0 \equiv -1 \).

Let us now formulate a sufficient condition for (epi).

\[ \text{Theorem 3.4} \]

\[ \text{Let} \] \( (D(f), g_0, g_1) \)

\[ \text{be transversal at} \] \( u \in D(f) \cap K \)

\[ \text{and set} \] \( \varphi = f + I_K \).

\[ \text{Then} \] \( \xi > \varphi(u) \)

\[ \text{implies that} \] \( |dG_\varphi|(u, \xi) = 1 \).

\[ \text{In particular, if} \) \( (D(f), g_0, g_1) \)

\[ \text{is transversal at} \] \( \text{any} u \in D(f) \cap K \), then \( \varphi \)

\[ \text{satisfies condition (epi)}. \]

The proof of this theorem is given below. It is based mainly on the following theorem, which is also fundamental for the investigation of the Palais-Smale condition in the next subsection. For

\[ u \in K \]

\[ \text{we use the notation} \]

\[ T_-(u) = \begin{cases} 
\{ v \in Y : g_0^0(u; v - u) \leq 0 \} & \text{if} \ g_0(u) < 0 \\
\{ v \in Y : g_0^0(u; v - u) \leq 0, g_1^0(u; v - u) \leq 0 \} & \text{if} \ g_0(u) = 0
\end{cases} \]

\[ T_+(u) = \begin{cases} 
\{ v \in Y : g_1^0(u; u - v) \leq 0 \} & \text{if} \ g_0(u) < 0 \\
\{ v \in Y : g_0^0(u; v - u) \leq 0, g_1^0(u; u - v) \leq 0 \} & \text{if} \ g_0(u) = 0
\end{cases}. \]
Theorem 3.5 Let \( u \in D(f) \cap K \). Assume that \((D(f), g_0, g_1)\) is transversal at \( u \) and set
\[
\forall v \in Y : \tilde{f}_1(v) = f_1(u) + f_1^0(u; v - u) .
\]

Then we have
\[
|d(f + I_K)| (u) = \min \left\{ |d \left( f_0 + \tilde{f}_1 + I_{T_- (u)} \right) (u) |, |d \left( f_0 + \tilde{f}_1 + I_{T_+ (u)} \right) (u) | \right\}
\]
and the right hand side is in turn equal to
\[
\min \left\{ \|f_0^* + f_1^* + \lambda_0 g_0^* + \lambda_1 g_1^*\| : f_0^* \in \partial f_0(u), f_1^* \in \partial f_1(u), \lambda_0 \geq 0, g_0^* \in \partial g_0(u), \lambda_1 \in \mathbb{R}, g_1^* \in \partial g_1(u), \lambda_0 g_0(u) = 0 \right\}
\]
if \( \partial f_0(u) \neq \emptyset \), otherwise is equal to \(+\infty\).

This theorem directly implies the following Lagrange Multiplier Rule, which can be used to derive the Euler-Lagrange equations in applications.

Corollary 3.6 Let \( u \in D(f) \cap K \) be a critical point of \( f + I_K \), i.e., \(|d(f + I_K)| (u) = 0\). If \((D(f), g_0, g_1)\) is transversal at \( u \), then \( \partial f_0(u) \neq \emptyset \) and there exist
\[
f_0^* \in \partial f_0(u), f_1^* \in \partial f_1(u), g_0^* \in \partial g_0(u), g_1^* \in \partial g_1(u), \lambda_0 \geq 0, \lambda_1 \in \mathbb{R}
\]
such that
\[
f_0^* + f_1^* + \lambda_0 g_0^* + \lambda_1 g_1^* = 0 \quad \text{and} \quad \lambda_0 g_0(u) = 0 .
\]

As a preparation for the proof of Theorem 3.5, we verify the following two Lemmata.

Lemma 3.7 Let \( C \) be a convex subset of \( Y \) and let \( u \in C \cap K \). If \((C, g_0, g_1)\) is transversal at \( u \), then we have
\[
0 \in \operatorname{int} (T_- (u) - C) \cap \operatorname{int} (T_+ (u) - C) .
\]

PROOF. Let us treat only the case \( g_0(u) = 0 \). Let \( u_- \in C \) be such that
\[
g_0^0 (u; u_- - u) < 0 , \quad g_1^0 (u; u_- - u) < 0
\]
and let \( \rho > 0 \) be such that
\[
\forall v \in B_\rho (u_-) : g_0^0 (u; v - u) < 0 , \quad g_1^0 (u; v - u) < 0 .
\]
Then it is easy to see that \( B_\rho (0) \subset (T_- (u) - C) \). In a similar way, it is possible to show that \( 0 \in \operatorname{int} (T_+ (u) - C) . \]

Lemma 3.8 Let \( u \in Y \) and let \( W \) be a compact subset of \( Y \). Then for every \( \varepsilon > 0 \) there exists \( \rho > 0 \) such that
\[
f_1(v + t(w - v)) - f_1(v) \leq tf_1^0 (u; w - v) + \varepsilon t
\]
whenever \( w \in W \), \( v \in B_\rho (u) \) and \( t \in [0, \rho] \).
Let $\delta > 0$.

Now let $\varepsilon > 0$ and let $\delta < \rho$ be such that $2\delta < \rho$.

Up to a subsequence, $(w_n)$ is convergent to some $w \in W$. Then, by the local Lipschitz continuity of $f_1$, it follows

$$\frac{f_1(v_n + t_n(w_n - v_n)) - f_1(v_n)}{t_n} > \frac{f_1^0(u; w_n - v_n) + \varepsilon t_n}{\delta}.$$  

Going to the upper limit as $n \to \infty$ and taking into account the continuity of $f_1^0(u; \cdot)$, we deduce that $f_1^0(u; w - u) > f_1^0(u; w - u) + \varepsilon$, which is a contradiction. \(\blacksquare\)

**Proof of Theorem 3.5.** Let us treat only the case $g_0(u) = 0$. Let

$$0 < \sigma < \min \left\{ \left| \frac{d\left(f_0 + \tilde{f}_1 + I_{T_-(u)}\right)}{d\left(f_0 + \tilde{f}_1\right)}(u) \right|, \left| \frac{d\left(f_0 + \tilde{f}_1 + I_{T_+(u)}\right)}{d\left(f_0 + \tilde{f}_1\right)}(u) \right| \right\}.$$  

Since $\left(f_0 + \tilde{f}_1 + I_{T_-(u)}\right)$ is lower semicontinuous and convex, by [24, Theorem (2.11)] we can find $w_- \in T_- (u)$ such that

$$f_0(w_-) + \tilde{f}_1(w_-) - f_0(u) + \tilde{f}_1(u) - \sigma \|w_- - u\|,$$

namely

$$f_0(w_-) - f_0(u) + f_1^0(u; w_- - u) < -\sigma \|w_- - u\|.$$  

On the other hand, by the transversality condition there exists $u_- \in D(f) = D\left(f_0 + \tilde{f}_1\right)$ such that $g_0^0(u; u_- - u) < 0$ and $g_1^0(u; u_- - u) < 0$. By substituting $w_-$ with $w_- + t(u_- - w_-)$ for some small $t \in [0, 1]$, we can therefore assume that $g_0^0(u; u_- - u) < 0$ and $g_1^0(u; u_- - u) < 0$.

In a similar way, we can find $w_+ \in T_+(u)$ such that

$$f_0(w_+) - f_0(u) + f_1^0(u; w_+ - u) < -\sigma \|w_+ - u\|,$$

$$g_0^0(u; w_+ - u) < 0, \quad g_1^0(u; u - w_+) < 0.$$  

Now let $\varepsilon > 0$ and let $W$ denote the convex hull of $\{w_-, w_+\}$. Taking into account Lemma 3.8, we can find $\rho \in [0, 1]$ such that

$$\forall v \in B_\rho(u) : f_0(w_-) - f_0(v) + f_1^0(u; w_- - v) < -\sigma \|w_- - v\|,$$

$$\forall v, z \in B_\rho(u) : g_0^0(z; w_- - v) < 0, \quad g_1^0(z; w_- - v) < 0,$$

$$\forall v \in B_\rho(u) : f_0(w_+) - f_0(v) + f_1^0(u; w_+ - v) < -\sigma \|w_+ - v\|,$$

$$\forall v, z \in B_\rho(u) : g_0^0(z; w_+ - v) < 0, \quad g_1^0(z; v - w_+) < 0,$$

$$\forall w \in W, \forall v \in B_\rho(u), \forall t \in [0, \rho] : f_1(v + t(w - v)) - f_1(v) \leq tf_1^0(u; w - v) + \varepsilon t.$$  

Let $\delta > 0$ be such that $2\delta < \rho < \rho \|w_- - u\|$ and $\delta < \rho \|w_+ - u\|$. For every $v \in K \cap B_\delta (u)$ and every $t \in [0, \delta]$ the function

$$\left\{ s \mapsto g_1 \left( v + \frac{s}{\|w_+ - v\|} (w_+ - v) + \frac{t - s}{\|w_- - v\|} (w_- - v) \right) \right\}$$
is strictly increasing on $[0, t]$ by Lebourg’s Theorem (see [19]). Moreover, it is negative for $s = 0$ and positive for $s = t$. Therefore there exists one and only one $s(v, t) \in [0, t]$ such that

$$g_1 \left( v + \frac{s(v, t)}{\|w_+ - v\|} (w_+ - v) + \frac{t - s(v, t)}{\|w_- - v\|} (w_- - v) \right) = 0$$

and it depends continuously on $(v, t)$.

Again by Lebourg’s Theorem, we have also

$$g_0 \left( v + \frac{s(v, t)}{\|w_+ - v\|} (w_+ - v) + \frac{t - s(v, t)}{\|w_- - v\|} (w_- - v) \right) \leq 0.$$  

Taking into account the convexity of $f_0$, we can define a continuous map

$$\mathcal{H} : (\mathcal{D} (f) \cap K \cap B_{\delta} (u)) \times [0, \delta] \to \mathcal{D} (f) \cap K$$

by

$$\mathcal{H}(v, t) = v + \frac{s(v, t)}{\|w_+ - v\|} (w_+ - v) + \frac{t - s(v, t)}{\|w_- - v\|} (w_- - v).$$

It is easy to check that $\|\mathcal{H}(v, t) - v\| \leq t$. Set

$$\alpha = \frac{s(v, t)}{\|w_+ - v\|}, \quad \beta = \frac{t - s(v, t)}{\|w_- - v\|}.$$

Since $\alpha + \beta \leq \rho$, we have

$$f(\mathcal{H}(v, t)) - f(v) =$$

$$= f_0 \left( v + \alpha (w_+ - v) + \beta (w_- - v) \right) - f_0(v) +$$

$$+ f_1 \left( v + (\alpha + \beta) \left( \frac{\alpha}{\alpha + \beta} w_+ + \frac{\beta}{\alpha + \beta} w_- - v \right) \right) - f_1(v) \leq$$

$$\leq \alpha (f_0(w_+) - f_0(v)) + \beta (f_0(w_-) - f_0(v)) +$$

$$+ (\alpha + \beta) f_1^0 \left( u; \frac{\alpha}{\alpha + \beta} (w_+ - v) + \frac{\beta}{\alpha + \beta} (w_- - v) \right) + \varepsilon (\alpha + \beta) \leq$$

$$\leq \frac{s(v, t)}{\|w_+ - v\|} (f_0(w_+) - f_0(v)) + f_1^0 (u; w_+ - v) +$$

$$+ \frac{t - s(v, t)}{\|w_- - v\|} (f_0(w_-) - f_0(v)) + f_1^0 (u; w_- - v) + \varepsilon (\alpha + \beta) \leq$$

$$\leq - \sigma s(v, t) - \sigma (t - s(v, t)) + \frac{\varepsilon t}{\min\{\|w_+ - v\|, \|w_- - v\|\}} =$$

$$= - \left( \sigma - \frac{\varepsilon}{\min\{\|w_+ - v\|, \|w_- - v\|\}} \right) t.$$

By Proposition 1.5 it follows

$$|d (f + I_K) |(u) \geq \sigma - \frac{\varepsilon}{\min\{\|w_+ - v\|, \|w_- - v\|\}},$$

hence $|d (f + I_K) |(u) \geq \sigma$, by the arbitrariness of $\varepsilon$. Finally, by the arbitrariness of $\sigma$, we deduce that

$$|d (f + I_K) |(u) \geq \min \left\{ |d (f_0 + \tilde{f}_1 + I_{T_-(u)}) |(u), |d (f_0 + \tilde{f}_1 + I_{T_+(u)}) |(u) \right\}.$$
To conclude the proof, it is sufficient to show that \( |d \left( f_0 + \hat{f}_1 + I_{T_-(u)} \right) | (u) \) is equal to

\[
\min \left\{ \| f_0^* + f_1^* + \lambda_0 g_0^* + \lambda_1 g_1^* \| : 
\begin{align*}
& f_0^* \in \partial f_0(u), f_1^* \in \partial f_1(u), \lambda_0 \geq 0, g_0^* \in \partial g_0(u), \lambda_1 \geq 0, g_1^* \in \partial g_1(u) 
\end{align*}
\right\}
\]

if \( \partial f_0(u) \neq \emptyset \), otherwise it is equal to +\( \infty \). Then for \( |d \left( f_0 + \hat{f}_1 + I_{T_-(u)} \right) | (u) \) a similar statement holds and the proof is complete.

Now, from [24, Theorem (2.11)] we know that \( |d \left( f_0 + \hat{f}_1 + I_{T_-(u)} \right) | (u) \) is equal to

\[
\min \left\{ \| \gamma^* \| : \gamma^* \in \partial \left( f_0 + \hat{f}_1 + I_{T_-(u)} \right) (u) \right\}
\]

if this set is not empty, otherwise it is +\( \infty \). On the other hand, we have \( D \left( \hat{f}_1 \right) = Y \) and \( 0 \in \text{int} \{ T_-(u) - D (f_0) \} \) by Lemma 3.7. From [12, Corollary 4.3.6] we deduce that

\[
\partial \left( f_0 + \hat{f}_1 + I_{T_-(u)} \right) (u) = \partial f_0(u) + \partial \hat{f}_1(u) + N_{T_-(u)}(u) = \partial f_0(u) + \partial f_1(u) + N_{T_-(u)}(u) .
\]

Again by the transversality condition, it is easy to show that

\[
0 \in \text{int} \left( \{ v \in Y : g_0^0(u; v - u) \leq 0 \} - \{ v \in Y : g_1^0(u; v - u) \leq 0 \} \right) .
\]

From [12, Theorem 4.1.16] and [19, Corollary 2.4.1] it follows

\[
N_{T_-(u)}(u) = \{ \lambda_0 g_0^* + \lambda_1 g_1^* : \lambda_0 \geq 0, g_0^* \in \partial g_0(u), \lambda_1 \geq 0, g_1^* \in \partial g_1(u) \} ,
\]

whence the assertion. \( \blacksquare \)

**Proof** of Theorem 3.4. Since \( f_1 \) is locally Lipschitz continuous, by Proposition 1.6 it is sufficient to treat the case \( f_1 = 0 \).

Define \( \hat{f} : Y \times \mathbb{R} \to \mathbb{R} \cup \{ +\infty \} \), \( \hat{g}_0, \hat{g}_1 : Y \times \mathbb{R} \to \mathbb{R} \) and \( \hat{K} \) by

\[
\hat{f}(v, \mu) = \begin{cases} 
\mathcal{G}_f(v, \mu) & \text{if } (v, \mu) \in \text{epi} (f) \\
+\infty & \text{if } (v, \mu) \notin \text{epi} (f) 
\end{cases} ,
\]

\[
\hat{g}_0(v, \mu) = g_0(v), \quad \hat{g}_1(v, \mu) = g_1(v) ,
\]

\[
\hat{K} = \{ (v, \mu) \in Y \times \mathbb{R} : \hat{g}_0(v, \mu) \leq 0, \hat{g}_1(v, \mu) = 0 \} = K \times \mathbb{R} .
\]

Then \( \hat{f} \) is lower semicontinuous and convex, \( \hat{g}_0, \hat{g}_1 \) are locally Lipschitz continuous and

\[
\left( \hat{f} + I_{\hat{K}} \right) (v, \mu) = \begin{cases} 
\mathcal{G}_\varphi(v, \mu) & \text{if } (v, \mu) \in \text{epi} (\varphi) \\
+\infty & \text{if } (v, \mu) \notin \text{epi} (\varphi) 
\end{cases} .
\]

Moreover it is easy to check that \( (\mathcal{D} (\hat{f}) , \hat{g}_0, \hat{g}_1) \) is transversal at any \( (u, \xi) \) with \( \xi \geq \varphi(u) \). By Theorem 3.5, for every \( \xi \geq \varphi(u) \) we have that

\[
|d \mathcal{G}_\varphi| (u, \xi) = |d \left( \hat{f} + I_{\hat{K}} \right) | (u, \xi) \geq
\]
\[ \geq \min \left\{ \left| d\left( \hat{f} + I_{\hat{T}_-(u,\xi)} \right) \right| (u,\xi), \left| d\left( \hat{f} + I_{\hat{T}_+(u,\xi)} \right) \right| (u,\xi) \right\}, \]

where \( \hat{T}_-(u,\xi) \) and \( \hat{T}_+(u,\xi) \) are related to \( \hat{K} \) in the obvious way. On the other hand, from [24, Theorem (2.11)] we deduce that
\[ \left| d\left( \hat{f} + I_{\hat{T}_-(u,\xi)} \right) \right| (u,\xi) = \min \left\{ \|\gamma^*\| : \gamma^* \in \partial \left( \hat{f} + I_{\hat{T}_-(u,\xi)} \right) (u,\xi) \right\}. \]

Of course we have \( (u,\varphi(u)) \in D \left( \hat{f} + I_{\hat{T}_-(u,\xi)} \right) \). If \( \xi > \varphi(u) \), it is then easy to show that \( \|\gamma^*\| \geq 1 \) for every \( \gamma^* \in \partial \left( \hat{f} + I_{\hat{T}_-(u,\xi)} \right) (u,\xi) \). It follows \( \left| d\left( \hat{f} + I_{\hat{T}_-(u,\xi)} \right) \right| (u,\xi) = 1 \). In a similar way, it is possible to prove that \( \left| d\left( \hat{f} + I_{\hat{T}_+(u,\xi)} \right) \right| (u,\xi) = 1 \), whence the assertion. \( \blacksquare \)

### 3.2 Palais-Smale condition

We now study the Palais-Smale condition. First we introduce some notions.

**Definition 3.9** We say that \( \partial f_0 \) is proper on bounded subsets, if for every (strongly) compact subset \( C \) of \( Y^* \) and every bounded subset \( B \) of \( Y \) the set
\[ \{ u \in D \left( f_0 \right) \cap B : \partial f_0(u) \cap C \neq \emptyset \} \]

has (strongly) compact closure in \( Y \).

**Definition 3.10** Let \( g : Y \rightarrow \mathbb{R} \) be a locally Lipschitz continuous function. We say that \( \partial g \) is compact, if for every bounded sequence \( (u_n) \) in \( Y \) and every sequence \( g_n^* \in \partial g(u_n) \), there exists a subsequence \( (g_{n_k})^{*} \) (strongly) convergent in \( Y^* \).

When \( g \) is of class \( C^1 \), this means that \( g' \) is compact (i.e. completely continuous) in the usual sense.

**Definition 3.11** Let \( c \in \mathbb{R} \). We say that \( (f,g_0,g_1) \) is uniformly transversal at level \( c \), if for every bounded sequence \( (u_n) \) in \( K \) with \( f(u_n) \rightarrow c \), there exist a subsequence \( (u_{n_k}) \) and \( u_-,u_+ \in D(f) \) such that

(a) if \( \lim \inf_{k} g_0(u_{n_k}) < 0 \), we have
\[ \lim \sup_{k} g_1^0(u_{n_k};u_- - u_{n_k}) < 0, \quad \lim \sup_{k} g_1^0(u_{n_k};u_{n_k} - u_+) < 0; \]

(b) if \( \lim_{k} g_0(u_{n_k}) = 0 \), we have
\[ \lim \sup_{k} g_0^0(u_{n_k};u_- - u_{n_k}) < 0, \quad \lim \sup_{k} g_0^0(u_{n_k};u_+ - u_{n_k}) < 0, \]
\[ \lim \sup_{k} g_1^0(u_{n_k};u_{n_k} - u_+) < 0. \]

We say that \( (f,g_1) \) is uniformly transversal at level \( c \), if \( (f,g_0,g_1) \) is uniformly transversal at level \( c \) for \( g_0 \equiv -1 \).
Definition 3.12 The function $f$ is said to be coercive on $Z \subset Y$, if for every $b \in \mathbb{R}$ the set $f^b \cap Z$ is bounded in $Y$.

Let us now formulate a criterion sufficient for condition $(PS)_c$.

**Theorem 3.13** Let $c \in \mathbb{R}$. Assume that $(f, g_0, g_1)$ is uniformly transversal at level $c$, that $\partial f_0$ is proper on bounded subsets and that $\partial f_1$, $\partial g_0$ and $\partial g_1$ are compact.

Then each bounded $(PS)_c$-sequence for $(f + I_K)$ admits a strongly convergent subsequence in $Y$. If, in addition, $f$ is coercive on $Z \subset K$, then $(f + I_K)$ satisfies $(PS)_c$ on $Z$.

**Proof.** Let $(u_n)$ be a bounded $(PS)_c$-sequence for $(f + I_K)$. Let $(u_{n_k})$, $u_-$ and $u_+$ be as in Definition 3.11. For the sake of simplicity, let us still denote $(u_{n_k})$ with $(u_n)$. Up to a further subsequence, we can suppose that $\lim_{n} g_0(u_n)$ exists in $[-\infty, 0]$. Then $(\mathcal{D}(f), g_0, g_1)$ is transversal at $u_n$ eventually as $n \to \infty$. Let us treat only the case $\lim_{n} g_0(u_n) = 0$. By Theorem 3.5 we can find $f_{0,n}^* \in \partial f_0(u_n)$, $f_{1,n}^* \in \partial f_1(u_n)$, $\lambda_0,n \geq 0$, $g_{0,n}^* \in \partial g_0(u_n)$, $\lambda_1,n \in \mathbb{R}$, $g_{1,n}^* \in \partial g_1(u_n)$ and $\gamma_n^* \in Y^*$ such that $\lambda_0,n g_0(u_n) = 0$, $\|\gamma_n^*\| \leq |d(f + I_K)(u_n)|$ and

\[
\lambda_0,n + f_{0,n}^* + \lambda_0,n g_{0,n}^* + \lambda_1,n g_{1,n}^* = \gamma_n^*.
\]

Up to a subsequence, we can suppose that $(f_{1,n}^*)$, $(g_{0,n}^*)$ and $(g_{1,n}^*)$ are strongly convergent in $Y^*$ and that $(\lambda_{1,n})$ has constant sign. If $\lambda_{1,n} \geq 0$, remark that

\[
f_0(u_-) - f_0(u_n) \geq \langle f_{0,n}^*, u_- - u_n \rangle = \langle \gamma_{0,n}^*, u_- - u_n \rangle - \langle f_{1,n}^*, u_- - u_n \rangle - \lambda_0,n \langle g_{0,n}^*, u_- - u_n \rangle - \lambda_1,n \langle g_{1,n}^*, u_- - u_n \rangle \geq \langle \gamma_{0,n}^*, u_- - u_n \rangle - \langle f_{1,n}^*, u_- - u_n \rangle - \lambda_0,n g_0^*(u_n; u_- - u_n) - \lambda_1,n g_1^*(u_n; u_- - u_n).
\]

By means of Lebourg’s Theorem, it is easy to show that $f_1$ is bounded on bounded subsets. Therefore $(f_0(u_n))$ is bounded. From the uniform transversality it follows

\[
sup_{n} \lambda_{0,n} < +\infty, \quad \sup_{n} \lambda_{1,n} < +\infty.
\]

If $\lambda_{1,n} \leq 0$, it is possible to show in a similar way that

\[
sup_{n} \lambda_{0,n} < +\infty, \quad \inf_{n} \lambda_{1,n} > -\infty.
\]

Therefore, up to a subsequence, $(\lambda_{0,n})$ and $(\lambda_{1,n})$ are convergent in $\mathbb{R}$. It follows that $(f_{0,n}^*)$ is strongly convergent in $Y^*$, whence the $(PS)_c$-sequence $(u_n)$ has a strongly convergent subsequence, as $\partial f_0$ is proper on bounded subsets.

The last assertion is evident, because every $(PS)_c$-sequence in $Z$ is bounded under the additional assumption $\gamma$. The next result may help in dealing with the assumptions of Theorem 3.13.

**Proposition 3.14** Let $Y$, $Z$ be two Banach spaces, $A : Y \to Z$ a map of class $C^1$ with $A'$ uniformly continuous on bounded subsets, $\gamma : Z \to \mathbb{R}$ a locally Lipschitz continuous function and let $g = \gamma \circ A$. Assume that $Y$ is reflexive and that $A$ is sequentially continuous from the weak to the strong topology.

Then the following facts hold:
(a) $\partial g$ is compact;

(b) for every $u,v \in Y$ we have $g^0(u,v) \leq \gamma^0(A(u);A'(u)v)$;

(c) the map $\{(u,v) \mapsto \gamma^0(A(u);A'(u)v)\}$ is sequentially upper semicontinuous with respect to the weak topology of $Y$.

**Proof.** In [19, Theorem 2.3.10] one can find property (b) and the formula

$$\forall u \in Y : \partial g(u) \subset A'(u)^* (\partial \gamma(A(u))).$$

If $(u_n)$ is a bounded sequence in $Y$, up to a subsequence $(u_n)$ is weakly convergent to some $u \in Y$. It follows that $(A'(u_n))$ is convergent to $A'(u)$ in the norm topology and that $A'(u)$ is compact. Since $(A(u_n))$ is strongly convergent to $A(u)$, the set

$$\bigcup_n \partial \gamma(A(u_n))$$

is bounded in $Y^*$. Then property (a) easily follows.

Finally, let $(u_n)$ and $(v_n)$ be two sequences in $Y$ weakly convergent to $u$ and $v$, respectively. Then $(A(u_n))$ is strongly convergent to $A(u)$ and $(A'(u_n)v_n)$ is strongly convergent to $A'(u)v$, whence assertion (c). $\blacksquare$

### 4 Rod theory

In this section we formulate the equations governing the planar equilibrium configurations of nonlinearly elastic rods in a specialized form sufficient for our applications. A more comprehensive presentation can be found in Antman [7] or Schuricht [42]. The presented theory is also called Cosserat or director theory of rods. Let $\{i,j,k\}$ be a fixed right-handed orthonormal basis in $\mathbb{R}^3$.

We identify the deformed rod with the region occupied by its intersection with the $\{i,j\}$-plane. The planar position field $p$ describing the deformed material points of the rod is assumed to have the form

$$p(s,\zeta) = r(s) + \zeta b(s) \text{ for } (s,\zeta) \in \Omega := [0,L] \times [-h(s),h(s)] \quad (4.1)$$

where $r, b$ are absolutely continuous mappings with values in the $\{i,j\}$-plane and $h$ is an upper-semicontinuous positive real function on $[0,L]$. Here $r$ describes the deformed configuration of some material curve in the rod, the so-called base curve (which is a curve of centroids in our special case), and $b(s)$ is a unit vector, called the director at $s$, describing the orientation of the cross-section at $s$. We interpret $s$ as length parameter and $\zeta$ as thickness parameter. Obviously, a planar configuration of a rod is uniquely determined by the pair $(r,b)$. We set $a := -k \times b$ and denote by $\theta$ the angle between $i$ and $a$ such that

$$a = \cos \theta i + \sin \theta j, \quad b = -\sin \theta i + \cos \theta j. \quad (4.2)$$

Thus a configuration can be alternatively described by $r$ and $\theta$. We define the strains $\xi = (\nu, \eta, \mu)$ for a configuration by

$$r' = \nu a + \eta b, \quad \mu := \theta'.$$
In the undeformed natural state, assumed to be straight, we take
\[ \nu = 1, \quad \eta = 0, \quad \mu = 0. \quad (4.3) \]
The requirement that the deformations be locally orientation-preserving can be expressed by the condition that
\[ \nu(s) > V(\mu(s), s) \text{ for } s \in [0, L], \quad \text{where} \quad V(\mu, s) := h(s)|\mu|. \quad (4.4) \]
Let us remark that this does not imply global injectivity. For given integrable functions \( \nu(\cdot), \eta(\cdot), \mu(\cdot), r_0 \in \mathbb{R}^2, \) and \( \theta_0 \in \mathbb{R}, \) we can represent a configuration by
\[ r(s) = r_0 + \int_0^s (\nu a + \eta b) \, dt, \]
\[ \theta(s) = 2\theta_0 + \int_0^s \mu \, dt. \quad (4.5) \]
By pure technical reasons we use the factor 2 in the last formula.

We set \( \Omega_s := \{(\tau, \zeta) \in \Omega : \tau \in [s, L]\}. \) In a deformed configuration, the material corresponding to \( \Omega_s \) exerts across section \( s \) a resultant contact force \( n(s) \) and a resultant contact couple \( m(s) \) on the material of \( \Omega \setminus \Omega_s. \) Naturally we have
\[ n(0) = 0 \quad \text{and} \quad m(0) = 0. \quad (4.7) \]
We assume that all other forces acting at the rod can be described by a finite vector-valued Borel measure \( \mathcal{P} \mapsto f(\mathcal{P}) \) assigning the resultant external force to the subbodies of the rod that correspond to Borel sets \( \mathcal{P} \subset \Omega. \) The force \( f \) causes the induced couple
\[ l_f(\mathcal{P}) := \int_{\mathcal{P}} \left( p(s, \zeta) - r(s) \right) \times df(s, \zeta) = \int_{\mathcal{P}} \zeta b(s) \times df(s, \zeta). \quad (4.8) \]
We now suppose that all couples which are different from \( m \) and \( l_f \) can be given by a finite vector-valued Borel measure \( \mathcal{P} \mapsto l(\mathcal{P}) \) which we call external couple. With the distribution functions
\[ f(s) := f(\Omega_s), \quad l_f(s) := l_f(\Omega_s), \quad l(s) := l(\Omega_s), \quad (4.9) \]
the equilibrium equations are given by (4.7) and
\[ n(s) - f(s) = 0 \quad \text{for } s \in [0, L], \quad \quad \quad (4.10) \]
\[ m(s) - \int_s^L r'(\tau) \times n(\tau) \, d\tau - l_f(s) - l(s) = 0 \quad \text{for } s \in [0, L]. \quad \quad \quad (4.11) \]
We introduce the so-called stress resultants \( (N, H, M) \) for a configuration by
\[ n = Na + Hb, \quad m = Mk. \]
The material of the rod is taken to be elastic, i.e., there are constitutive functions \( \tilde{N}, \tilde{H}, \tilde{M} \) such that the stress resultants are determined by the strains through
\[ N = \tilde{N}(\nu, \eta, \mu, s), \quad H = \tilde{H}(\nu, \eta, \mu, s), \quad M = \tilde{M}(\nu, \eta, \mu, s). \quad (4.12) \]
The domain of these functions is restricted obviously by (4.4). We assume that \( \hat{N}, \hat{H}, \hat{M} \) are continuously differentiable with respect to \( \xi = (\nu, \eta, \mu) \) and that the Jacobian
\[
J := \begin{pmatrix}
\hat{N}_\nu & \hat{N}_\eta & \hat{N}_\mu \\
\hat{H}_\nu & \hat{H}_\eta & \hat{H}_\mu \\
\hat{M}_\nu & \hat{M}_\eta & \hat{M}_\mu
\end{pmatrix}
\]
is positive-definite. (4.13)

We also assume that
\[
\begin{align*}
\hat{N}(\nu, \eta, \mu, s) & \to \begin{cases} +\infty \\ -\infty \end{cases} & \nu \to +\infty \\
\hat{H}(\nu, \eta, \mu, s) & \to \pm \infty & \eta \to \pm \infty, \\
\hat{M}(\nu, \eta, \mu, s) & \to \pm \infty & \mu \text{ approaches its positive and negative}
\end{align*}
\]extremes of the region (4.4). (4.14)

Since the undeformed configuration of the rod is assumed to be straight and the base curve is a curve of centroids, we can adopt the symmetry conditions
\[
\hat{N}(\nu, \cdot, \mu, s), \hat{M}(\nu, \cdot, \mu, s) \text{ are even}, \quad \hat{H}(\nu, \cdot, \mu, s) \text{ is odd},
\]
(4.17)
\[
\hat{N}(\nu, \eta, \cdot, s), \hat{H}(\nu, \eta, \cdot, s) \text{ are even}, \quad \hat{M}(\nu, \eta, \cdot, s) \text{ is odd}
\]
(4.18)
(cf. Antman & Marlow [8]).

For a large class of materials the matrix \( J \) in (4.13) is symmetric. In this case the material is called hyperelastic and there exists a real-valued function \( W \) of \( (\nu, \eta, \mu, s) \), the so-called stored energy function, such that
\[
\hat{N} = W_\nu, \quad \hat{H} = W_\eta, \quad \hat{M} = W_\mu.
\]
(4.19)

The domain of definition for \( W \) is given by (4.4). Let us suppose the natural condition that the stored energy \( W \) approaches infinity under complete compression, i.e.,
\[
W(\nu, \eta, \mu, s) \to \infty \quad \text{as} \quad \nu - V(\mu, s) \to 0.
\]
(4.20)

The growth conditions (4.14), (4.15), (4.16) obviously imply that \( W \) tends to infinity as the strains \( (\nu, \eta, \mu) \) tend to infinity. Observe that \( W(\cdot, \cdot, \cdot, s) \) is strictly convex by (4.13). The symmetry conditions (4.17) and (4.18) give that
\[
W(\nu, \cdot, \cdot, s), \quad W(\nu, \cdot, \cdot, s) \text{ are even.}
\]
(4.21)

The total stored energy of the rod is the functional
\[
E_s(\nu, \eta, \mu) = \int_0^L W(\nu(s), \eta(s), \mu(s), s) \, ds.
\]

Since we use variational methods, we restrict our attention to hyperelastic materials.

In our applications we study buckling problems subject to a terminal load \(-\lambda i, \lambda \in \mathbb{R}, \) acting at the point \((s, \zeta) = (L, 0)\), i.e., we have a prescribed external force
\[
f_0(P) := \begin{cases} 
-\lambda i & \text{if } (L, 0) \in P, \\
0 & \text{otherwise.}
\end{cases}
\]
(4.22)
This force is conservative and has the potential energy

\[ E_p(p) := -\int_\Omega p(s, \zeta) \cdot df_0(s, \zeta) = \lambda r(L) \cdot i. \]  

(4.23)

In Sections 5 and 6 we verify the existence of critical points of the total energy \( E_s + E_p \) with respect to certain constraints using the presented abstract critical point theory. We shall see that these critical points satisfy certain Euler-Lagrange equations which are equivalent to the equilibrium conditions (4.7), (4.10), (4.11).

**Note.** Our analysis describing buckling covers more general constitutive laws than presented in the previous rod theory and in other previous investigations of buckling in the literature. We can handle stored energy functions \( W \) that are strictly convex in \((\nu, \eta, \mu)\) and satisfy some coercivity condition. We do not need further differentiability properties. In this general case the constitutive relations in (4.19) do not make sense. We rather have to replace the constitutive functions \( \hat{N}, \hat{H}, \hat{M} \) by the set-valued mapping

\[ (\nu, \eta, \mu, s) \mapsto (\hat{N}, \hat{H}, \hat{M})^T := \partial_\xi W, \]  

(4.24)

where \( \partial_\xi W \) denotes the subdifferential of \( W \) with respect to \( \xi \). Thus lack of differentiability of \( W \) causes a multiplicity in the constitutive law, i.e., for certain strains \((\nu, \eta, \mu)\) the corresponding stress resultants \((N, H, M)\) are not uniquely determined which models certain plastic effects.

To avoid technicalities we do not claim to emphasize this generalization and to work out all corresponding details for the rod theory, though they are straightforward. Nevertheless our buckling results in the next sections cover such more general situations, both for free buckling and for rods restricted by obstacles.

5 Buckling of free rods

5.1 Formulation of the problem and results

Here we study the free buckling of a rod subjected to a terminal load of the form (4.22) and some boundary conditions by variational methods.

Let us fix the left end of the rod, i.e., we assume that

\[ r(0) = 0. \]

By (4.5), (4.6), a configuration of the rod is then determined by

\[ u := (\nu, \eta, \mu, \theta_0) = (\xi, \theta_0) \]

where we choose

\[ u \in X := L^{p_1} \times L^{p_2} \times L^{p_3} \times S^1, \quad p_1, p_2, p_3 > 1. \]

Here \( L^p \) denotes the usual Lebesgue space of \( p \)-integrable functions on \([0, L]\) and \( S^1 \) denotes the one-dimensional sphere. Recalling (4.6) we see that \( \theta_0 \in S^1 \) induces some multiplicity which, however, is wanted by technical reasons. By \( r[u](s), \theta[u](s), \) etc., we indicate that the values are computed for the configuration determined by \( u \). We now suppose that the point \((s, \zeta) = (L, 0)\) at the right end can only move along a line through the origin, i.e., we demand that

\[ g_1(u) := r[u](L) \cdot j = 0. \]  

(5.1)
Furthermore we apply a terminal load $-\lambda i$ at this point $(L, 0)$ (cf. (4.22)). With the parameter $\lambda$ we get the total energy of the rod

$$E_\lambda(u) := E_s(u) + E_p(u) = \int_0^L W[u](s) \, ds + \lambda r[u](L) \cdot i.$$ 

We are interested in critical points of $E_\lambda$ in $X$ with respect to the constraint (5.1) for given parameters $\lambda \in \mathbb{R}$. Setting $g_0(u) = -1$ on $X$ we have in the notation of Section 3 that

$$K = \{ u \in X : g_1(u) = 0 \}. \quad (5.2)$$

By technical purposes we set $W(\nu, \eta, \mu, \cdot, \cdot, \cdot, \cdot, s) := +\infty$ outside the domain of definition given by $\nu - V(\mu, s) > 0$. Thus our problem is equivalent to the determination of critical points of the functional $(E_\lambda + I_K)$ in $X$. We claim to apply Theorem 2.5 for this purpose.

For our analysis, $W(\cdot, \cdot, \cdot, s)$ is assumed to be strictly convex for all $s \in [0, L]$ and $W(\nu, \eta, \mu, \cdot, \cdot, \cdot, \cdot)$ be measurable for all $(\nu, \eta, \mu)$. Further differentiability properties, as used in the rod theory introduced in the previous section, are not invoked. Conditions (4.20), (4.21) be satisfied and, in addition, we assume the standard coercivity condition that

$$W(\nu, \eta, \mu, s) \geq c(\|\nu\|^{p_1} + \|\eta\|^{p_2} + \|\mu\|^{p_3}) + \gamma(s) \quad (5.3)$$

where $c > 0$ is a constant and $\gamma$ an integrable function. Observe that $W(\cdot, \cdot, \cdot, s)$ is lower semicontinuous.

We define $\Phi : X \rightarrow X$ by

$$\Phi(u) = \Phi(\nu, \eta, \mu, \theta_0) := (\nu, -\eta, -\mu, -\theta_0). \quad (5.4)$$

$\Phi$ is obviously an isometry with $\Phi^2 = Id$ and

$$\text{Fix}(X) = \{ u \in X | u = (\nu, 0, 0, 0) \text{ or } u = (\nu, 0, 0, \pi) \}.$$

Obviously $(\nu, 0, 0, 0)$ and $(\nu, 0, 0, \pi)$ yield geometrically identical states of the rod with $\theta(0) = 0$. Observe that $(\nu, 0, 0, \pm \frac{\pi}{2})$ provide corresponding reflected states with $\theta(0) = \pi$ which are also geometrically identical. We call states of the form $(\nu, 0, 0, \frac{k\pi}{2})$, $k \in \mathbb{Z}$, trivial states. Clearly such states respect the constraint (5.1). Let us mention that without factor 2 in (4.6) the set $\text{Fix}(X)$ would contain trivial states with $\theta(0) = \pi$ which would bother our analysis. Recalling (4.21), (4.5), (4.6), we get the following symmetry properties for the action of $\Phi$.

**Lemma 5.5** We have that

$$W[\Phi(u)](s) = W[u](s), \quad \theta[\Phi(u)](s) = -\theta[u](s),$$

$$r[\Phi(u)](s) \cdot i = r[u](s) \cdot i, \quad r[\Phi(u)](s) \cdot j = -r[u](s) \cdot j,$$

$$g_1(u) = g_1(\Phi(u)).$$

Consequently, $E_s$, $E_p$, and $E_\lambda$ are $\Phi$-invariant. Moreover, $g_1(u) = 0$ implies that $g_1(\Phi(u)) = 0$, i.e., constraint (5.1) is invariant under $\Phi$. 

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By Lemma 5.5 we can expect pairs \((u, \Phi(u))\) of critical points.

**Theorem 5.6** Assume that \(E_\lambda \neq +\infty\). Then:

(i) For every \(\lambda \in \mathbb{R}\) there exist positive functions \(\nu_\lambda, \tilde{\nu}_\lambda \in L^p\) such that \(E_\lambda\) is finite at

\[
u_\lambda^0 := (\nu_\lambda(s), 0, 0, 0) \in X \quad \text{and} \quad \nu_\lambda^{\pm \pi} := (\tilde{\nu}_\lambda(s), 0, 0, \pm \frac{\pi}{2}) \in X\]  

(5.7)

and \(\nu_\lambda^0, \nu_\lambda^{\pm \pi}\) are critical points of \(E_\lambda\) and of \(E_\lambda + I_K\). \(\nu_\lambda, \tilde{\nu}_\lambda\) are determined by the relations

\[ (-\lambda, 0, 0) \in \partial_\nu W(\nu_\lambda(s), 0, 0, s), \quad (\lambda, 0, 0) \in \partial_\nu W(\tilde{\nu}_\lambda(s), 0, 0, s), \]  

(5.8)

and \(\nu_\lambda(s) \in [0, 1]\) for \(\lambda \geq 0\). Moreover \(\nu_\lambda^0, \nu_\lambda^{\pm \pi}\) are the only critical points of \(E_\lambda\) and of \(E_\lambda + I_K\) in the set of trivial states. Clearly \(\nu_\lambda^{\pm \pi}\) correspond to geometrically identical configurations and \(\nu_\lambda^0 \in \text{Fix}(X)\). We call \(\nu_\lambda^0, \nu_\lambda^{\pm \pi}\) trivial solutions according to \(\lambda\).

(ii) Suppose that there is some interval \([s_a, s_b] \subset [0, L]\) and some \(\varepsilon > 0\) with \(\max_{s_a \leq s \leq s_b} h(s) < \frac{1}{\varepsilon}\) such that

\[
\frac{\int_{s_a}^{s_b} \left( W(\nu_\lambda(s), \varepsilon \nu_\lambda(s), \varepsilon \nu_\lambda(s), s) - W(\nu_\lambda(s), 0, 0, s) \right) ds}{\lambda \left( \int_{s_a}^{s_b} \nu_\lambda(s) \, ds \right)^2} \to 0 \quad \text{as} \quad \lambda \to \infty.
\]  

(5.9)

Then for every \(k \in \mathbb{N}\) there is \(\lambda_k > 0\) such that for all \(\lambda > \lambda_k\) the function \(E_\lambda\) has at least \(k\) distinct pairs \((u_1, \Phi(u_1)), \ldots, (u_k, \Phi(u_k))\) of critical points with respect to the constraint (5.1) and \(E_\lambda(u_j) < \inf \{E_\lambda(u) | u \in \text{Fix}(X)\}\).

(iii) Assertion (ii) remains true if we replace (5.9) with

\[
\frac{\int_{s_a}^{s_b} \left( W(\nu_\lambda(s), 0, \varepsilon \nu_\lambda(s), s) - W(\nu_\lambda(s), 0, 0, s) \right) ds}{\lambda \left( \int_{s_a}^{s_b} \nu_\lambda(s) \, ds \right)^3} \to 0 \quad \text{as} \quad \lambda \to \infty.
\]  

(5.10)

**Corollary 5.11** Suppose that \(W\) does not depend on \(s\) and that it is twice continuously differentiable in the other arguments. Then (5.9) can be replaced with

\[
\frac{1}{\lambda} \max_{|\nu|, |\mu| \leq \varepsilon \nu_\lambda} \left\| \frac{\partial^2 W(\nu_\lambda, \eta, \mu)}{\partial (\eta, \mu)^2} \right\| \to 0 \quad \text{as} \quad \lambda \to \infty.
\]  

(5.12)

**Remark 5.13** (1) The theorem shows that buckling can be ensured by certain constitutive assumptions. Since the limit in (5.9) seems to be essentially determined by the behavior of \(W\) in \(\nu\) and \(\mu\), assertion (iii) is probably of more theoretical interest and it roughly shows that variations in \(\mu\) only lead to a stronger condition. Observe that conditions (5.9) and (5.10) do not need differentiability and, in so far, we generalize buckling results of Antman & Rosenfeld [11].

(2) The fact that the constitutive conditions (5.9) and (5.10) must be studied only on some subinterval \([s_a, s_b] \subset [0, L]\) means that it is sufficient for the buckling of the whole rod if a separated small part of the rod would already buckle.

(3) In the case where \(W\) is twice continuously differentiable in the arguments \(\nu, \eta, \mu\) and satisfies the constitutive condition (5.9) or (5.10) we can expect that the trivial solution \(u_\lambda^0\) is not a local minimizer of the energy \(E_\lambda\) for \(\lambda > \lambda_1\). This means, roughly speaking, that the trivial
solutions are not stable under large compression. If however $W$ is not smooth, then this must not be the case. Though stability is not the subject of this paper let us still mention that there exist materials where the trivial solution $u^0_\lambda$ is stable for all compression forces (cf. Antman & Pierce [10]).

(4) The thickness $h(\cdot)$ of the rod can be considered as a stiffness parameter of the material, i.e., we have included in the stored energy $W$ some dependence on $h$. Usually we have that the numerator in (5.9) or in (5.10) approaches zero if the maximal thickness $h_0 := \max\{h(s) : s \in [0, L]\}$ goes to zero (cf. Antman & Rosenfeld [11, p.540] and Antman & Pierce [10]). In this case we can argue analogously as in our proof to obtain that for any $\lambda > 0$ and $k \in \mathbb{N}$ we can find a sufficiently small $h_k^0 > 0$ such that all rods with a maximal thickness $h_0 < h_k^0$ have at least $k$ distinct pairs of nontrivial solutions under the load $\lambda$.

(5) By the used methods we can study also other buckling problems where, e.g., the constraint $r(L) \cdot j = 0$ is replaced by $\theta(0) = 0$ or $\theta(L) = 0$ or combinations of it. The most significant difference then will be the determination of suitable variations to verify transversality. However the presented ideas should be a sufficient stimulation to handle such other cases. Observe that, in contrast to the considered constraint (5.1), the “angle-constraints” are linear.

(6) The growth condition (5.3) ensures coercivity for $E_\lambda$ which is necessary for existence results. The strict convexity of $W(\cdot, \cdot, \cdot, s)$ combined with (5.3) ensures that $\partial E_\lambda$ is proper on bounded subsets (cf. Definition 3.9) which finally is used to verify the Palais-Smale condition for $E_\lambda$ (cf. Theorem 3.13).

We can now ask whether the detected critical points, satisfying the very abstract condition that the weak slope equals zero, are physically reasonable. The next theorem shows that the Euler-Lagrange equations hold.

**Theorem 5.14** Let $u \in X$ be any critical point determined in Theorem 5.6. Then there exist $\lambda_1 \in \mathbb{R}$ and measurable functions $s \mapsto N(s), H(s), M(s)$ with

$$(N(s), H(s), M(s)) \in \partial W(\nu(s), \eta(s), \mu(s), s) \quad \text{a.e. on } [0, L]$$

such that $\lambda_1 r(L) = 0$ and

$$n[u](s) = -\lambda_1 i - \lambda_1 j, \quad m[u](s) = \lambda r(s) \times i - \lambda_1 (r(L) - r(s)) \times j \quad \text{a.e. on } [0, L],$$

where

$$n[u](s) := N(s)a(\theta(s)) + H(s)b(\theta(s)), \quad m[u](s) := M(s)k$$

and $r(s) = r[u](s)$ (observe (4.2), (5.24)).

**Remark 5.18** (1) We readily verify that the equations (5.16) represent the equilibrium conditions (4.10), (4.11) for our buckling problem and (5.15) agrees with (4.19) or (4.24), i.e., our solutions have physical sense in the light of the general rod theory. This finally justifies our approach based on the abstract notion of the weak slope.

(2) The equations (5.16) can be taken as basis for further regularity investigations. E.g., for homogeneous material and a twice continuously differentiable stored energy $W$ our critical points
\( u \) correspond to configurations with twice continuously differentiable functions \( r[u](\cdot) \) and \( \theta[u](\cdot) \). We can see this by standard arguments observing that the inversion of the constitutive equations (4.12) is continuously differentiable (cf. also Schuricht [43]). Such questions, however, are not the subject of this paper.

(3) Condition \( \lambda_1 r(L) = 0 \) expresses some natural singularity in our buckling problem for solutions with \( r(L) = 0 \). Roughly speaking, we have the following situation. If we rotate any such solution with \( r(L) = 0 \) around the origin, then we obtain a continuum of configurations which all satisfy the side condition (5.1) and the equilibrium equations, but which correspond to different \( \lambda \)'s. For a simpler rod model a careful investigation of this effect combined with stability arguments can be found in Maddocks [30]. We however do not pause to study this point in more detail. Such situations can be prevented, e.g., when we prescribe the angle \( \theta_0 \). Remark, however, that our multiplicity result is not affected by that point and we obtain \( k \) equilibria all with the same \( \lambda \).

(4) Using Proposition 3.1 and (5.60) we observe that a solution with \( r(L) \neq 0 \) is even a free critical point of \( E_\lambda \) belonging to \( K \). We can however expect that the stability behavior with respect to \( X \) and to \( K \) is different.

Before we proof the theorems let us still give a special example for the stored energy \( W \) that shows that the conditions of Theorem 5.6 can be really satisfied and, on the other hand, that they must not be fulfilled for all materials. Let \( p_1, p_2, p_3 > 1 \) and \( 0 < q < \infty \). We set

\[
W(\nu, \eta, \mu, s) := \nu^{p_1} + |\eta|^{p_2} + |\mu|^{p_3} + \frac{1}{\nu^q} + \frac{1}{\nu^2 - h^2(s)\mu^2} + (q + 2 - p_1)\nu
\]  \hspace{1cm} (5.19)

where \( \nu > h(s)|\mu| \), and \( W(\nu, \eta, \mu, s) = \infty \) elsewhere.

**Proposition 5.20** Let \( W \) be given as in (5.19). Then the stored energy \( W \) satisfies all constitutive assumptions of the rod theory introduced in Section 4, it fulfils the coercivity condition (5.3), and \( E_\lambda \neq +\infty \). In addition, let \( [s_a, s_b] \subset [0, L] \) be any subinterval and let \( \varepsilon > 0 \) be such that \( \max_{s_a \leq s \leq s_b} h(s) < \frac{1}{\varepsilon} \).

(a) If \( q > 3 \), then (5.9) is satisfied.

(b) If \( q = 3 \), then the left expression in (5.9) is bounded as \( \lambda \to \infty \).

(c) If \( q < 3 \) and \( h(s) \geq h_1 \) on \( [s_a, s_b] \) for some \( h_1 > 0 \), then the left expression in (5.9) goes to infinity as \( \lambda \to \infty \).

**Corollary 5.21** The assertion of Proposition 5.20 remains true if we add the nonsmooth convex expression \( c_1|\eta| + c_2|\mu| \), \( c_1, c_2 \geq 0 \), to the energy \( W \) in (5.19).

**Remark 5.22** (1) In formula (5.19) the first three terms on the right hand side ensure that \( W \) is strictly convex, the fifth term accounts for the singularity caused by orientation preservation, the last term merely ensures that the undeformed state is stress free, and the fourth term determines the buckling behavior. Roughly speaking, for a small number \( q \) axial compression of the rod is relatively soft and we can expect that the compressed trivial straight solution remains stable for all axial loads. An increase of \( q \), however, makes compression harder and harder and for large \( q \) buckling occurs for sufficiently large loads.
(2) The corollary demonstrates that we can verify buckling also if $W$ is actually nonsmooth at the trivial solution.

5.2 Proofs

Let us prepare the proof of Theorem 5.6 by some preliminary investigations. We claim to apply the abstract results of Sections 2 and 3. In the light of Proposition 2.9 and the arguments in front of it we introduce the Banach space

$$Y := \mathcal{L}^{p_1} \times \mathcal{L}^{p_2} \times \mathcal{L}^{p_3} \times \mathbb{R}.$$ 

We obviously have that

$$Y^* = \mathcal{L}^{p_1^*} \times \mathcal{L}^{p_2^*} \times \mathcal{L}^{p_3^*} \times \mathbb{R}$$

where $1/p_i + 1/p_i^* = 1$. Observe that all important mappings of our problem defined on the metric space $X$ can be extended in natural way to the Banach space $Y$. If we apply Proposition 2.9, then we use the mapping

$$Y \ni u = (\nu, \eta, \mu, \theta_0) \mapsto p(u) := (\nu, \eta, \mu, \hat{\theta}_0) \in X$$

(5.23)

where $\hat{\theta}_0$ is the obvious “projection” onto $S^1$. $p$ is locally an isometry.

In view of (4.6) and without danger of confusion we shall use the notation

$$\theta(s) = \theta[u](s) = 2\theta_0 + \int_0^s \mu(\tau) \, d\tau.$$ 

(5.24)

For $\hat{u} \in X$ or $\hat{u} \in Y$ we use

$$\hat{u} = (\hat{\nu}, \hat{\eta}, \hat{\mu}, \hat{\theta}_0) = (\xi, \hat{\theta}_0), \quad \hat{\theta}(s) = \theta[\hat{u}](s) = 2\hat{\theta}_0 + \int_0^s \hat{\mu} \, d\tau$$

and, analogously, we furnish all symbols corresponding to $u_n$ with a subscript $n$ etc. For $u^* \in Y^*$ let us write

$$u^* = (\nu^*, \eta^*, \mu^*, \theta_0^*) = (\xi^*, \theta_0^*).$$

Let us introduce some terminology used by Rockafellar (cf. Rockafellar [37], Aubin & Frankowska [13]):

Definition 5.25 .

(i) Let $Z$ be a topological space. A multifunction $\Gamma : \mathbb{R} \rightrightarrows 2^Z$ that is closed-valued, i.e., $\Gamma(s)$ are closed subsets of $Z$, is called measurable if for each closed set $C \subset Z$ the set

$$\Gamma^{-1}(C) := \{s \in \mathbb{R} | \Gamma(s) \cap C \neq \emptyset \}$$

is Lebesgue measurable.

(ii) We say that $W$ is a normal integrand if $W(\cdot, \cdot, \cdot, s)$ is lower semicontinuous and the multifunction

$$[0, L] \ni s \mapsto \text{epi} W(\cdot, \cdot, \cdot, s) = \{(\nu, \eta, \mu, \kappa) \in \mathbb{R}^3 \times \mathbb{R} | W(\nu, \eta, \mu, s) \leq \kappa \}$$

is measurable. (Observe that $\text{epi} W(\cdot, \cdot, \cdot, s)$ is closed for all $s$ by the lower semicontinuity.)
Lemma 5.26.

(i) $W$ is a normal integrand.

(ii) $E_s$ is convex on $Y$ and the multifunction

$$s \mapsto \text{graph } \partial W(\cdot, \cdot, \cdot, s) := \{((\xi, \xi^\ast)) | \xi^\ast \in \partial \xi W(\xi, s)\}$$

(5.27)

is measurable.

(iii) If $E_s(u) \neq +\infty$, then for $u \in Y$ the subdifferential $\partial E_s(u) \subset Y^*$ is given by

$$\partial E_s(u) = \{(\xi^\ast, 0) \in Y^* | \xi^\ast(s) \in \partial \xi W(\nu(s), \eta(s), \mu(s), s) \text{ a.e. on } [0, L]\}.$$ 

(5.28)

(iv) $E_s$ is bounded below and weakly lower semicontinuous on $Y$.

(v) $\partial E_s$ is proper on bounded subsets.

(vi) $E_\lambda$ is bounded below on $Y$ and it is coercive on

$$Y_0 := \{u = (\nu, \eta, \mu, \theta_0) \in Y | \theta_0 \in [0, 2\pi]\}.$$ 

Proof. Obviously the domain of $W(\cdot, \cdot, \cdot, s)$ has nonempty interior for every $s \in [0, L]$ (cf. (4.4)). Then by Rockafellar [37, Coroll. 2E] $W$ is a normal integrand. The convexity of $E_s$ on $Y$ is evident. The measurability of the multifunction given in (5.27) is a consequence of a result going back to Attouch (cf. Rockafellar [37, Theorem 2W]). Formula (5.28) follows by Rockafellar [37, Coroll. 3E].

The boundedness below of $E_s$ is evident by (5.3). A result of Rockafellar implies the weak lower semicontinuity of $E_s$ on $Y$ (cf. Rockafellar [37, Coroll. 3D] and the considerations surrounding it). Let us mention that the results of Rockafellar applied in this proof can be found partially also in Barbu & Precupanu [14, p. 116].

We consider (v). Let $C \subset Y^*$ be compact and $B \subset Y$ bounded. We choose a sequence

$$\{u_j\}_{j=1}^{\infty} \subset \{v \in \mathcal{D}(E_s) \cap B | \partial E_s(v) \cap C \neq \emptyset\}.$$ 

and, accordingly, $u^*_k \in \partial E_s(u_k) \cap C$ for $k \in \mathbb{N}$. Since $B$ is bounded and $C$ compact, up to a subsequence we can assume that

$$u_k \rightharpoonup u \in Y, \quad u^*_k \rightharpoonup u^* \in C \subset Y^*.$$ 

(5.29)

Furthermore we can suppose, at least for a subsequence, that

$$\xi^*_k(s) = (\nu^*_k(s), \eta^*_k(s), \mu^*_k(s)) \rightharpoonup (\nu^*(s), \eta^*(s), \mu^*(s)) = \xi^*(s) \text{ a.e. on } [0, L].$$ 

(5.30)

The strict convexity of $W(\cdot, \cdot, \cdot, s)$ and formula (5.28) imply the pointwise convergence

$$\xi_k(s) = (\nu_k(s), \eta_k(s), \mu_k(s)) \rightharpoonup (\nu(s), \eta(s), \mu(s)) = \xi(s) \text{ a.e. on } [0, L].$$ 

(5.31)

Observe that the weak and the pointwise limit coincide in $\mathcal{L}^p$ (cf. Zeidler [48, p. 1023]). Using (5.3) and the convexity of $W(\cdot, \cdot, \cdot, s)$ we get

$$c|\nu_k(s)|^p + \gamma(s) \leq W(\xi_k(s), s) \leq W(1, 0, 0, s) - (\xi^*_k(s), (1, 0, 0) - \xi_k(s)) \text{ a.e. on } [0, L].$$
Analogous relations hold with $|\eta_k(s)|^{p_2}$ and $|\mu_k(s)|^{p_3}$ instead of $|\nu_k(s)|^{p_1}$ on the left hand side. By (5.29), (5.30), and (5.31), Lebesgue’s Dominated Convergence Theorem implies that

$$\|u_k\| \to \|u\|.$$  

Since $Y$ is a uniformly convex space, $u_k \to u$. This verifies the assertion.

We finally show (vi). With the notation $\|\cdot\|_p$ for the $L^p$-norm, relation (5.3) and the continuous embeddings $L^p \hookrightarrow L^1$, $p > 1$, yield that

$$E_\lambda(u) = \int_0^L \left(W(\nu(s), \mu(s), s) - \lambda(\nu(s) \cos \theta(s) - \eta(s) \sin \theta(s))\right) ds$$

$$\geq \int_0^L \left[c(\|\nu\|_{p_1}^{p_1} + |\Delta\eta(s)|^{p_2} + |\mu(s)|^{p_3}) + \gamma(s) - \lambda(|\nu| + |\eta(s)|)\right] ds$$

$$\geq c(\|\nu\|_{p_1}^{p_1} + |\mu|_{p_2}^{p_2} + |\eta|_{p_3}^{p_3}) + c_0 - \lambda\|\nu\|_{1} + |\eta|_{1})$$

$$\geq \|\nu\|_{p_1}(c(\|\nu\|_{p_1}^{p_1} - c_1\lambda) + |\eta|_{p_2}(c\|\eta\|_{p_2}^{p_2} - c_2\lambda) + \|\mu\|_{p_3}^{p_3} + c_0$$

where $c_0, c_1, c_2$ are suitable real numbers. This way we readily see the assertion. 

**Lemma 5.32**

(i) We have that

$$g_1(u) = \int_0^L (\nu \sin \theta + \eta \cos \theta) ds, \quad E_p(u) = \lambda \int_0^L (\nu \cos \theta - \eta \sin \theta) ds. \quad (5.33)$$

(ii) The functionals $g_1$, $E_p$ are weakly continuous and have continuous derivatives on $Y$ with

$$\langle g_1'(u), \hat{\tilde{u}} \rangle = \int_0^L (\hat{\tilde{\nu}} \sin \theta + \hat{\tilde{\eta}} \cos \theta + \hat{\tilde{\eta}} (\nu \cos \theta - \eta \sin \theta) ds, \quad (5.34)$$

$$g_1'(u) = (\sin \theta, \cos \theta, (r(L) - r(\cdot)) \cdot i, 2r(L) \cdot i), \quad (5.35)$$

$$\langle E'_p(u), \hat{\tilde{u}} \rangle = \lambda \int_0^L (\hat{\tilde{\nu}} \cos \theta - \hat{\tilde{\eta}} \sin \theta - \hat{\tilde{\eta}} (\nu \sin \theta + \eta \cos \theta) ds, \quad (5.36)$$

$$E'_p(u) = (\lambda \cos \theta, -\lambda \sin \theta, -\lambda(r(L) - r(\cdot)) \cdot j, -2\lambda r(L) \cdot j). \quad (5.37)$$

(iii) $g_1' : Y \to Y^*$ is strongly continuous, i.e.,

$$u_n \to u \implies g_1'(u_n) \to g_1'(u).$$

(iv) The generalized gradients $\partial g_1$, $\partial E_p$ are compact and $\partial g_1(u) = \{g_1'(u)\}$, $\partial E_p(u) = \{E'_p(u)\}$.  

**Proof.** By (4.5), (4.23), (5.1) assertion (i) is evident. Now we choose a weak convergent sequence $u_n \to u$ in $Y$, i.e., $\nu_n \to \nu$, $\eta_n \to \eta$, $\mu_n \to \mu$, and $\theta_{0,n} \to \theta_0$ in the corresponding spaces. By (5.24), $\theta$ and $\theta_n := \theta[u_n]$ belong to the Sobolev space $W^{1,p_3}$ of $L^{p_3}$-functions with generalized $p_3$-integrable derivative. Using the compact embedding of $W^{1,p_3}$ into the space $C$ of continuous
functions we get the uniform convergence \( \theta_n(s) \Rightarrow \theta(s) \) on \([0, L]\). But this yields the strong convergences

\[
\theta_n \to \theta, \quad \sin \theta_n \to \sin \theta, \quad \cos \theta_n \to \cos \theta
\]  

(5.38)
in \( L^p \) for any \( p > 1 \). Observing the special form of \( g_1 \) and \( E_p \) given in (5.33), this in particular implies the weak continuity of \( g_1, E_p \). A straightforward computation yields the existence of the derivatives \( g'_1, E'_p \) and formulas (5.34), (5.36). Applying Fubini’s Theorem and observing (4.5), we get

\[
\int_0^L \hat{\theta} (\nu \cos \theta - \eta \sin \theta) \, ds = \int_0^L \left( \nu(s) \cos \theta(s) - \eta(s) \sin \theta(s) \right) \left( 2 \hat{\theta}_0 + \int_0^s \hat{\mu}(\tau) \, d\tau \right) \, ds
\]

\[
= 2 \hat{\theta}_0 \nu(L) \cdot i + \int_0^L \hat{\mu}(\tau) \int_0^\tau \left( \nu(s) \cos \theta(s) - \eta(s) \sin \theta(s) \right) \, ds \, d\tau
\]

\[
= 2 \hat{\theta}_0 \nu(L) \cdot i + \int_0^L \hat{\mu}(s) \left( r(L) - r(s) \right) \cdot i \, ds.
\]  

(5.39)

Thus in view of (5.34)

\[
g'_1(u) = \left( \sin \theta, \cos \theta, (r(L) - r(\cdot)) \cdot i, 2r(L) \cdot i \right) \in Y^*.
\]  

(5.40)

By arguments analogous to that yielding (5.38), the weak convergence \( u_n \to u \) in \( Y \) implies the strong convergence

\[
r_n \cdot i \to r \cdot i \quad \text{in} \quad L^p, \quad r_n(L) \cdot i \to r(L) \cdot i \quad \text{in} \quad \mathbb{R}
\]

for any \( p > 1 \). This way we obtain the strong continuity and also the (normal) continuity of \( g'_1 \). Analogously these properties hold for \( E'_p \) and we have that

\[
E'_p(u) = \left( \lambda \cos \theta, -\lambda \sin \theta, -\lambda (r(L) - r(\cdot)) \cdot j, -2\lambda r(L) \cdot j \right) \in Y^*.
\]

Clarke’s calculus tells us that \( \partial g_1(u) = \{ g'_1(u) \} \) and \( \partial E_p(u) = \{ E'_p(u) \} \) in this case (cf. Clarke [19]). The compactness of \( \partial g_1(u) \) and \( \partial E_p(u) \) is then evident by the strong continuity of \( g'_1 \) and \( E'_p \), respectively. \[\blacksquare\]

**Lemma 5.41** We have that

(i) \( (D(E_\lambda), g_1) \) is transversal at any \( u \in D(E_\lambda) \subset Y \) satisfying (5.1).

(ii) \( (E_\lambda, g_1) \) is uniformly transversal at any level \( c \in \mathbb{R} \) in \( Y \).

**Proof.** Choose \( u \in D(E_\lambda) \) with \( g_1(u) = 0 \). We first suppose that \( \cos \theta(s) = 0 \) on \([0, L]\) for \( \theta(s) = \theta[u](s) \). Then \( \nu(s) = 0 \) a.e. on \([0, L]\) by (5.33) and by \( g_1(u) = 0 \). But this contradicts the finiteness of \( E_\lambda(u) \) by (4.20) (cf. also (4.4)). Thus by the continuity of \( \theta(s) \) there exists an interval \( \mathcal{I}_\theta \subset [0, L] \) with nonzero measure and

\[
\cos \theta(s) \neq 0 \quad \text{on} \quad \mathcal{I}_\theta.
\]  

(5.42)

For \( k \in \mathbb{N} \), we set

\[
\mathcal{I}_k := \{ s \in [0, L] : W(\nu(s), \eta(s) - 1, \mu(s), s) \leq k \text{ and } W(\nu(s), \eta(s) + 1, \mu(s), s) \leq k \}.
\]  

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The set $\mathcal{I}_k$ is obviously measurable. By the convexity of $W(\cdot,\cdot,\cdot,s)$ we have that
\begin{equation}
W(\nu(s),\eta(s) + \delta, \mu(s), s) \leq k \quad \text{for all } \delta \in [-1, 1], \ s \in \mathcal{I}_k. \tag{5.43}
\end{equation}

Obviously $\mathcal{I}_k \subset \mathcal{I}_{k+1}$. Since $E_\lambda(u)$ is finite, $[0, L] \cup \bigcup_{k=1}^{\infty} \mathcal{I}_k$ must have measure zero. Therefore we can find some $k_0 \in \mathbb{N}$ such that $\tilde{\mathcal{I}} := \mathcal{I}_{k_0} \cap \mathcal{I}_\theta$ has nonzero measure. With
\[
\tilde{\theta}(s) := \begin{cases}
\theta(s) & \text{for } s \in \tilde{\mathcal{I}}, \\
\frac{\pi}{2} & \text{otherwise},
\end{cases}
\]
we introduce
\[
u_+ := u + \tilde{u}, \quad \nu_- := u - \tilde{u} \quad \text{where } \tilde{u} := (0, \cos \tilde{\theta}, 0, 0). \tag{5.44}
\]

Obviously $\cos \tilde{\theta} \in L^p$ and therefore $\tilde{u} \in Y$. By (5.43) we readily see that $u_+, u_- \in D(E_s)$ which is equivalent to $u_+, u_- \in D(E_\lambda)$.

Recalling (5.40), (5.42), we obtain
\[
g_\lambda'(u; u_- - u) = -\langle g_\lambda'(u), \tilde{u} \rangle = -\int_\tilde{\mathcal{I}} (\cos \theta(s))^2 ds < 0.
\]

Since $u - u_+ = -\tilde{u}$, we have also that $g_\lambda'(u; u - u_+) < 0$ and assertion (i) follows.

Let us now choose any $c \in \mathbb{R}$ and let $(u_n)$ be a bounded sequence in $Y$ with $g_\lambda(u_n) = 0$ and $E_\lambda(u_n) \to c$. Since $Y$ is reflexive, there exists a subsequence, denoted the same way, which weakly converges to some $u \in Y$. $E_\lambda$ is weakly lower semicontinuous, because $E_s$ has this property and $E_p$ is weakly continuous by Lemma 5.32. Hence $E_\lambda(u) \leq c$. Since $E_\lambda$ is bounded below, $u \in D(E_\lambda)$. Furthermore $g_\lambda(u) = 0$, since $g_\lambda$ is weakly continuous by Lemma 5.32. For $u$ we can now repeat the procedure from the proof of assertion (i), i.e., we define $u_+, u_-$ as in (5.44) and obtain that $u_+, u_- \in D(E_\lambda)$. Since $g_\lambda'$ is strongly continuous by Lemma 5.32, this finally yields
\[
\lim_{n \to \infty} g_\lambda'(u_n; u_- - u_n) = \lim_{n \to \infty} \langle g_\lambda'(u_n), u_- - u_n \rangle = \langle g_\lambda'(u), u_- - u \rangle
\]
\[
= -\langle g_\lambda'(u), \tilde{u} \rangle < 0
\]
and, analogously, $\lim_{n \to \infty} g_\lambda'(u_n; u_n - u_+) < 0$. This verifies assertion (ii).

**Proof** of Theorem 5.6. Recalling (5.33) we see that for trivial states $u \in X$
\[
E_\lambda(u) = \int_0^L \left( W(\nu(s), 0, 0, s) \pm \lambda \nu(s) \right) ds. \tag{5.45}
\]

For $\lambda \in \mathbb{R}$ let us study the problems
\[
\min_{(\nu, \eta, \mu) \in \mathbb{R}^3} W(\nu, \eta, \mu, s) \pm \lambda \nu \quad \text{for } s \in [0, L]. \tag{5.46}
\]

Recall that $W$ is continued by $+\infty$ outside the effective domain. Since the functions to be minimized are strictly convex on the effective domain and even in $\eta$ and $\mu$, for every $s \in [0, L]$ we obtain unique minimizers of the form $(\nu_\lambda^\pm(s), 0, 0)$ and
\[
0 \in \partial_k \left( W(\nu_\lambda^\pm(s), 0, 0, s) \pm \lambda \nu_\lambda^\pm(s) \right) = \partial_k W(\nu_\lambda^\pm(s), 0, 0, s) \pm (\lambda, 0, 0). \tag{5.47}
\]

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By technical reasons we have taken the minimum over the closed set \( \mathbb{R}^3 \), but we readily see that the solutions actually satisfy \( \nu_\chi^+(s) - V(0,s) = \nu_\chi^+(s) > 0 \). Clearly \( \nu_{\chi}^+ = \nu_{\chi}^- \). Thus for the most arguments we can restrict our attention to \( \nu_\chi^+ \). We set \( \nu_\chi := \nu_\chi^+ \) and \( \tilde{\nu}_\chi := \nu_\chi^- \).

By Lemma 5.26 the function \( W \) is a normal integrand and \( W(\nu, \eta, \mu, s) := \lambda \nu \) is obviously also a normal integrand. Hence the sum \( W(\nu, \eta, \mu, s) + \lambda \nu \) is again a normal integrand (cf. Rockafellar [37, Prop. 2M]). A result of Rockafellar now says that the multifunction assigning the set of minimizers of (5.46) to each \( s \) is measurable (cf. Rockafellar [37, Theorem 2K]). The uniqueness of the minimizers gives the measurability of the function \( s \mapsto \nu_\chi(s) \).

(5.3) and (5.47) imply that

\[
c|\nu_\chi(s)|^{p_1} + \gamma(s) \leq W(\nu_\chi(s), 0, 0, s) \leq W(1, 0, 0, s) + \gamma(1 - \nu_\chi(s)).
\]

(5.48)

Let \( c_1 \in \mathbb{R} \) be such that \( c_1 \leq \frac{c}{2} |t|^{p_1} + \lambda t \) for all \( t \in \mathbb{R} \). Then

\[
\frac{c}{2} |\nu_\chi(s)|^{p_1} + c_1 \leq c|\nu_\chi(s)|^{p_1} + \lambda \nu_\chi(s) \leq W(1, 0, 0, s) - \gamma(s) + \lambda.
\]

(5.49)

Since we assumed the natural undeformed state of the rod to be straight, the strictly convex function \( W(\cdot, \cdot, s) \) has its minimum at \((1,0,0)\) (cf. (4.3)). This in particular means that \( W(1,0,0,\cdot) \) is integrable on \([0,L]\), because otherwise \( E_s \equiv +\infty \). Thus the right hand side in (5.49) is an integrable function and \( \nu_\chi \in L^{p_1} \). Coming back to (5.48), we see also that \( W(\nu_\chi(\cdot), 0, 0, \cdot) \) is integrable for all \( \lambda \in \mathbb{R} \), i.e., \( E_\lambda(u_0^0) \) and \( E_\lambda(u_\pm) \) are finite. (5.48) and the minimality of \((1,0,0)\) in particular show that \( \nu_\chi(s) \leq 1 \) when \( \lambda \geq 0 \).

(5.28), (5.37), and (5.47) yield that

\[
(-\lambda, 0, 0, 0) \in \partial E_s(u_0^0), \quad E_p^\gamma(u_\lambda^0) = (\lambda, 0, 0, 0).
\]

Thus by Proposition 3.1

\[
|dE_\lambda|(u_0^0) = 0, \quad |d(E_\lambda + I_K)|(u_0^0) = 0.
\]

Analogously we can conclude for \( u_\pm^\lambda \). Observe that it is the same to compute these weak slopes in the metric space \( X \) or in the Banach space \( Y \).

Let us now assume that some trivial state

\[
\tilde{u}_\lambda = (\tilde{\nu}_\chi(s), 0, 0, k\pi), \quad k \in \mathbb{Z},
\]

is a critical point of \((E_\lambda + I_K)\), i.e., \(|d(E_\lambda + I_K)|(\tilde{u}_\lambda) = 0\). Lemma 5.41 and Theorem 3.5 give us that

\[
0 = \min \{|E_s^* + E_p^\gamma(\tilde{u}_\lambda) + \lambda_1 g_1'(\tilde{u}_\lambda)| : E_s^* \in \partial E_s(\tilde{u}_\lambda), \quad \lambda_1 \in \mathbb{R} \}.
\]

(5.50)

We readily get that \(|r(\tilde{u}_\lambda)|(s) \cdot j = 0 \) on \([0,L]\) and \(|r(\tilde{u}_\lambda)|(L) \cdot i \neq 0 \). Using (5.28), (5.35), and (5.37), we see that (5.50) can only be fulfilled with \( \lambda_1 = 0 \) and we obtain the necessary condition

\[
(-\lambda, 0, 0, 0) \in \partial E_s(\tilde{u}_\lambda),
\]

i.e.,

\[
(-\lambda, 0, 0, 0) \in \partial_2 W(\tilde{\nu}_\chi(s), 0, 0), \quad \text{a.e. on } [0,L].
\]

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But the strict convexity of \( W(\cdot, \cdot, s) \) and (5.8) yield that \( \bar{\nu}_\lambda(s) = \nu_\lambda(s) \) a.e. on \([0, L]\) and this means uniqueness. We argue analogously for trivial states \((\nu, 0, 0, k \pi)\), \(k\) odd, and for critical points of \( E_\lambda \). This way we have verified assertion (i).

Let us now consider assertion (ii). We claim to apply Theorem 2.5 to the energy \( E_\lambda \) on the metric space \( K \) given in (5.2) or, equivalently, we apply Theorem 2.5 to the functional \( f_\lambda := E_\lambda + I_K \) on \( X \). By Lemma 5.26 and Lemma 5.32 the energy \( E_\lambda \) is lower semicontinuous. Because \( K \) is closed, \( f_\lambda \) is also lower semicontinuous. \( E_\lambda \) and \( K \) are \( \Phi \)-invariant by Lemma 5.5. Hence \( f_\lambda \) is \( \Phi \)-invariant. \( f_\lambda \) satisfies condition (epi) by Theorem 3.4, Proposition 2.9 and Lemma 5.41 (cf. (5.23)). By Lemma 5.26 the functional \( f_\lambda \) is bounded below. Obviously \( E_\lambda = E_s + E_p \). \( \partial E_s \) is proper on bounded subsets by Lemma 5.26, \( \partial E_p, \partial g_1 \) are compact by Lemma 5.32 and \( E_\lambda \) is coercive on \( Y_0 \) by Lemma 5.26. \((E_\lambda, g_1)\) is uniformly transversal at any level \( c \in \mathbb{R} \) by Lemma 5.41. Since \( p(Y_0) = X \) for \( p \) given in (5.23), the functional \( f_\lambda \) satisfies condition \((PS)_c\) on \( X \) by Theorem 3.13 and Proposition 2.9 (observe that \( \partial g_0 \) is compact for \( g_0 \equiv -1 \)). We shall show below that for every \( k \in \mathbb{N} \) there is a number \( \lambda_k > 0 \) such that for each \( \lambda > \lambda_k \) we can find a continuous \( \Phi \)-equivariant map \( \psi_k^\lambda : S^{k-1} \to K \) with

\[
\sup_{x \in S^{k-1}} E_\lambda(\psi_k^\lambda(x)) < \inf_{u \in \text{Fix}(K)} E_\lambda(u). \tag{5.51}
\]

This then verifies the assertion.

**Note.** Let us shortly explain how we just at this point take advantage of the factor 2 in formula (4.6). For the construction of the map \( \psi_k^\lambda \) we claim to use certain small perturbations of \( u_0^k \) where \( E_\lambda \) decreases. Thus we need that the right hand side of (5.51) is not smaller than \( E_\lambda(u_0^k) \). This is essentially ensured by that factor 2, because the infimum is attained at \( u_0^k \) in that case. Without that device (5.51) cannot be verified in general, since from the mechanical point of view \( u_0^k \) are expected to be global minimizers. We alternatively could overcome the difficulties by distinguishing between different components of \( \text{Fix}(K) \). But this would cause other technicalities.

In order to compose \( \psi_k^\lambda \) we start with the construction of some special variations of the trivial solution \( u_0^k \) in \( X \). Let \( s_a = s_0 < \tau_1 < s_1 < \ldots < s_k < \tau_{k+1} < s_{k+1} = s_b \) be such that

\[
\kappa_\lambda := \frac{1}{2(k+1)} \int_{s_a}^{s_b} \nu_\lambda(s)\, ds = \int_{s_{i-1}}^{s_i} \nu_\lambda(s)\, ds = \int_{s_i}^{s_{i+1}} \nu_\lambda(s)\, ds = \frac{1}{2} \int_{s_{i-1}}^{s_i} \nu_\lambda(s)\, ds.
\]

\[
\eta_i(s) := \begin{cases} 
\varepsilon \nu_\lambda(s) & \text{on } [s_{i-1}, s_i], \\
0 & \text{otherwise},
\end{cases} \quad i = 1, \ldots, k+1,
\]

\[
\mu_i(s) := \begin{cases} 
\varepsilon \nu_\lambda(s) & \text{on } [s_{i-1}, \tau_i], \\
-\varepsilon \nu_\lambda(s) & \text{on } [\tau_i, s_i], \\
0 & \text{otherwise},
\end{cases} \quad \theta_i(s) := \int_0^s \mu_i(\tau)\, d\tau, \quad i = 1, \ldots, k. \tag{5.52}
\]

We easily verify that

\[
\theta_i(s) \begin{cases} 
= 0 & \text{for } s \not\in [s_{i-1}, s_i] \\
\geq 0 & \text{on } [s_{i-1}, s_i]
\end{cases}, \quad \theta_i(\tau_i) = \varepsilon \kappa_\lambda, \quad i = 1, \ldots, k.
\]

Using the fact that \((\frac{1}{3} \theta_i^2)' = \pm \varepsilon \nu_\lambda \theta_i^2\) and \((\frac{1}{2} \theta_i^2)' = \pm \varepsilon \nu_\lambda \theta_i\), we obtain

\[
\frac{\varepsilon^2}{2} \kappa_\lambda^3 = \frac{\varepsilon}{2} \theta_i^3(s)_{s_{i-1}}^{\tau_i} = \int_{s_{i-1}}^{\tau_i} \nu_\lambda(s) \theta_i(s)^2\, ds = \int_{s_i}^{s_{i+1}} \nu_\lambda(s) \theta_i(s)^2\, ds,
\]

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\[
\varepsilon_2^2 = \frac{1}{2} \theta_i^2(s) = \int_{s_{i-1}}^{s_i} \varepsilon \nu_\lambda(s) \theta_i(s) \, ds = \int_{s_{i-1}}^{s_i} \varepsilon \nu_\lambda(s) \theta_i(s) \, ds .
\] (5.53)

Let us set
\[
u_i := (0, \eta_i, \mu_i, 0) \in \mathbf{Y}, \quad i = 1, \ldots, k, \quad \nu_{k+1} := (0, \eta_{k+1}, 0, 0) \in \mathbf{Y}.
\]

Obviously the \( \nu_i \) depend on \( \lambda \). But since \( \nu_\lambda(s) \leq 1 \) for \( \lambda \geq 0 \), we have that
\[
0 \leq \theta_i(s) \leq \varepsilon \kappa_\lambda \leq \frac{\varepsilon (s_b - s_a)}{2(k + 1)} \quad \text{for all } s \in [0, L], \quad \lambda \geq 0.
\]

Therefore we can find \( \rho \in (0, 1) \) independently of \( \lambda \geq 0 \) such that
\[
\cos(t \theta_i(s)) \leq 1 - \frac{1}{2} t^2 \theta_i(s)^2, \quad \left| \sin(t \theta_i(s)) \right| \geq \frac{1}{2} |t| \theta_i(s) \quad \text{for } s \in [0, L], \quad |t| \leq \rho .
\] (5.54)

For \( x = (x_1, \ldots, x_k) \in \mathbb{R}^k \), \( \tau \in \mathbb{R} \) we consider the equation
\[
\tau \left( \nu_0 + \sum_{i=1}^k x_i \nu_i + \nu_{k+1} \right) = 0 .
\] (5.55)

By (5.33) this is equivalent to
\[
\sum_{i=1}^k \int_{s_{i-1}}^{s_i} \left( \nu_\lambda \sin x_i \theta_i + x_i \varepsilon \nu_\lambda \cos x_i \theta_i \right) \, ds + \int_{s_k}^{s_{k+1}} \tau \varepsilon \nu_\lambda \, ds = 0.
\]

Hence (5.55) is satisfied by the unique continuous resolution
\[
\tau = \tilde{\tau}(x) := - \frac{1}{2 \varepsilon \kappa_\lambda} \sum_{i=1}^k \int_{s_{i-1}}^{s_i} \left( \nu_\lambda \sin x_i \theta_i + x_i \varepsilon \nu_\lambda \cos x_i \theta_i \right) \, ds
\]

for all \( x \in \mathbb{R}^k \). We readily see that \( \tilde{\tau}(-x) = -\tilde{\tau}(x) \). Observe that \( \tilde{\tau} \) depends on \( \lambda \). However, independently of \( \lambda \geq 0 \) we can find some \( \sigma > 0 \) such that
\[
|\tilde{\tau}(tx)| \leq \frac{1}{2 \varepsilon \kappa_\lambda} \sum_{i=1}^k \int_{s_{i-1}}^{s_i} (|t| \nu_\lambda \theta_i + |t| \varepsilon \nu_\lambda) \, ds = \frac{(2 + \kappa_\lambda)k}{2} |t| < 1 \quad \text{for } x \in S^{k-1}, \quad |t| \leq \sigma
\]
(cf. (5.53), (5.54)).

Choose now \( \bar{t} := \min\{\rho, \sigma\} < 1 \). By the convexity and the symmetry of \( W \) we have that
\[
W(\nu_\lambda(s), \bar{t} x_i \varepsilon \nu_\lambda(s), \bar{t} x_i \varepsilon \nu_\lambda(s), s)
\leq W(\nu_\lambda(s), \varepsilon \nu_\lambda(s), \varepsilon \nu_\lambda(s), s) \quad \text{for } x \in S^{k-1},
\] (5.56)
\[
W(\nu_\lambda(s), 0, \tau \varepsilon \nu_\lambda(s), s)
\leq \frac{1}{2} \left( W(\nu_\lambda(s), \tau \varepsilon \nu_\lambda(s), \tau \varepsilon \nu_\lambda(s), s) + W(\nu_\lambda(s), -\tau \varepsilon \nu_\lambda(s), \tau \varepsilon \nu_\lambda(s), s) \right)
= W(\nu_\lambda(s), \tau \varepsilon \nu_\lambda(s), \tau \varepsilon \nu_\lambda(s), s)
\leq W(\nu_\lambda(s), \varepsilon \nu_\lambda(s), \varepsilon \nu_\lambda(s), s) \quad \text{for } \tau \in [0, 1].
\] (5.57)

We define
\[
\psi_k^\lambda(x) := u^0_\lambda + \bar{t} \sum_{i=1}^k x_i u_i + \tilde{\tau}(\bar{t}x) u_{k+1}.
\] (5.58)
By our previous investigations we readily see that $\psi^\lambda_k$ maps $S^{k-1}$ into $K$, i.e., $\psi^\lambda_k(x) \in X$ and $g_1(\psi^\lambda_k(x)) = 0$. Furthermore $\psi^\lambda_k$ is continuous and, by the antisymmetry of $\check{\tau}$, it is $\Phi$-equivariant.

By (5.56), (5.57) we readily get
\[
E_s(\psi^\lambda_k(x)) - E_s(u^0_k) \leq \int_{s_a}^{s_b} \left( W(\nu(\lambda)(s), \nu(\lambda)(s), \varepsilon \nu(\lambda)(s), s) - W(\nu(\lambda)(s), 0, 0, s) \right) ds.
\]
Using (5.33) and (5.54) we can estimate
\[
\text{Using (5.33) and (5.54) we can estimate}
\]
\[
E_p(\psi^\lambda_k(x)) - E_p(u^0_k) = E_p(u^0_k + \int_1^k x_i u_i + \check{\tau}(t \lambda) u_{k+1}) - E_p(u^0_k)
\]
\[
= \lambda \sum_{i=1}^k \int_{s_i}^{s_{i+1}} \left[ \nu(\lambda)(s) \left( \cos(\check{\tau}(x, \lambda)(s)) - 1 \right) - \check{\tau}(t \lambda) \sin(\check{\tau}(x, \lambda)(s)) \right] ds
\]
\[
\leq -\lambda \sum_{i=1}^k \int_{s_i}^{s_{i+1}} \left[ \frac{1}{2} \check{\tau}^2 x_i^2 \nu(\lambda)(s) \theta_i(s)^2 + \frac{1}{2} \check{\tau}^2 x_i^2 \sin(\check{\tau}(x, \lambda)(s)) \right] ds
\]
\[
= -\lambda \frac{\check{\tau}^2}{4} \sum_{i=1}^k x_i^2 \int_{s_i}^{s_{i+1}} \nu(\lambda)(s) \theta_i(s)^2 + 2 \check{\tau}^2 \sin(\check{\tau}(x, \lambda)(s)) ds
\]
\[
= -\lambda \frac{\check{\tau}^2}{4} \left( \frac{2 \check{\tau}^2}{3} \kappa_\lambda^2 + 2 \check{\tau}^2 \kappa_\lambda \right).
\]

By assumption (5.9) there exist some $\lambda_k \geq 0$ and $c_\lambda > 0$ such that $E_\lambda(\psi^\lambda_k(x)) < E_\lambda(u^0_k) - c_\lambda$ for all $x \in S^{k-1}$. Since $E_\lambda(u^0_k) \leq E_\lambda(u)$ for all $u \in \text{Fix } X$ by (5.45) and (5.46), we readily verify (5.51) and assertion (ii).

For the proof of assertion (iii) we argue similarly to the previous case. We construct $s_0, \ldots, s_{k+1}, \tau_1, \ldots, \tau_{k+1}, \mu_1, \ldots, \mu_k$ as before and, in addition, $\mu_{k+1}$ analogously to (5.52) with respect to the intervals $[s_k, \tau_{k+1}], [\tau_{k+1}, s_{k+1}]$. We now study variations of the form
\[
u_i := (0, 0, 0, 0) \in Y, \quad i = 1, \ldots, k + 1.
\]
Equation (5.55) has then the form
\[
\hat{g}(x, \tau) := \sum_{i=1}^k \int_{s_i}^{s_{i+1}} \nu_\lambda \sin x_i \theta_i ds + \int_{s_k}^{s_{k+1}} \nu_\lambda \sin \tau \theta_{k+1} ds = 0.
\]
Let $|\tau| \leq \rho$ and $\|x\| \leq \rho/4k$. By (5.53) and (5.54) we can estimate
\[
\left| \sum_{i=1}^k \int_{s_i}^{s_{i+1}} \nu_\lambda \sin x_i \theta_i ds \right| \leq \sum_{i=1}^k \left| \int_{s_i}^{s_{i+1}} \nu_\lambda |x_i| \theta_i ds \right| \leq \frac{\rho}{4} \kappa_\lambda^2,
\]
\[
\left| \int_{s_k}^{s_{k+1}} \nu_\lambda \sin \tau \theta_{k+1} ds \right| \geq \int_{s_k}^{s_{k+1}} \frac{\rho}{2} \nu_\lambda |\tau| \theta_{k+1} ds = \frac{1}{2} |\tau| \kappa_\lambda^2.
\]
Consequently
\[
\hat{g}(x, \rho) > 0, \quad \hat{g}(x, -\rho) < 0 \text{ for } \|x\| \leq \frac{\rho}{4k}.
\]
Thus, by the continuity of $\hat{g}$, for each $x$ with $\|x\| \leq \frac{\rho}{4k}$ we can find some $\check{\tau}(x) \in [-\rho, \rho]$ such that $\hat{g}(x, \check{\tau}(x)) = 0$. Since $\rho$ can be assumed to be so small that $\cos(\rho \theta_{k+1}(s)) > 0$ on $[0, L]$, we have by (5.34) that
\[
\frac{d}{d\tau} \hat{g}(x, \check{\tau}(x)) = \int_{s_k}^{s_{k+1}} \theta_{k+1} \nu \cos(\check{\tau}(x)\theta_{k+1}) ds > 0.
\]
Therefore, by the Implicit Function Theorem, $\tilde{\tau}$ is the unique resolution of (5.59) for $\|x\| \leq \rho/4k$. Obviously we can suppose that $\rho < 1$. Then we get with $\tilde{t} := \rho/4k$ and $\psi_k^\lambda(x)$ according to (5.58) by the same arguments as in the previous case that

$$E_s(\psi_k^\lambda(x)) - E_s(u_\lambda^0) \leq \int_{s_0}^{s_k} \left( W(\nu_\lambda(s), 0, \varepsilon \nu \lambda(s), s) - W(\nu_\lambda(s), 0, 0, s) \right) ds,$$

$$E_p(\psi_k^\lambda(x)) - E_p(u_\lambda^0) \leq -\frac{\tilde{t}^2}{6} \varepsilon^2 \kappa_\lambda^2.$$

This then readily implies the assertion. $\blacksquare$

**Proof** of Corollary 5.11. By the Taylor Theorem there exists $\rho_\lambda \in ]0,1[$ such that

$$W(\nu_\lambda, \varepsilon \nu_\lambda, \varepsilon \nu_\lambda) - W(\nu_\lambda, 0, 0) = \frac{1}{2} \varepsilon^2 W(\nu_\lambda, \rho_\lambda \varepsilon \nu_\lambda, \rho_\lambda \varepsilon \nu_\lambda) \left( \varepsilon \nu_\lambda \right) \left( \varepsilon \nu_\lambda \right) \leq \max_{|\eta|,|\mu| \leq \varepsilon \nu_\lambda} \left\| \frac{\partial^2 W(\nu_\lambda, \eta, \mu)}{\partial \eta^2} \right\| \varepsilon^2 \nu_\lambda^2.$$

This way we readily see that (5.9) is implied by (5.12). $\blacksquare$

**Proof** of Theorem 5.14. Let $u$ be a critical point provided by Theorem 5.6. Then $E_s(u)$ is obviously finite and $(D(E_\lambda), g_1)$ is transversal at $u$ by Lemma 5.41. Thus we can apply Corollary 3.6. By (5.35), (5.37), (4.2), (5.1) we get

$$E_p'(u) = \left( \lambda a(\theta(\cdot)) \cdot i, \lambda b(\theta(\cdot)) \cdot i, -\lambda (r[u](\cdot) \times i) \cdot k, 0 \right),$$

$$g_1'(u) = \left( a(\theta(\cdot)) \cdot j, b(\theta(\cdot)) \cdot j, (r[u](L) \times r[u](\cdot)) \times j) \cdot k, 2(r[u](L) \times j) \cdot k \right).$$

Hence by Corollary 3.6 and Lemma 5.26 we can find $\lambda_1 \in \mathbb{R}$ and measurable functions $N, H, M$ satisfying (5.15) such that

$$\left( N(s), H(s), M(s), 0 \right) + E_p'(u) + \lambda_1 g_1'(u) = 0 \text{ in } Y^*.$$  \hspace{1cm} (5.60)

With the notation of (5.17) this readily yields (5.16). The last component in (5.60) still gives that $\lambda_1 r[u](L) \times j = 0$. Since $r(L) \times j \neq 0$ for $r(L) \neq 0$, this implies $\lambda_1 r(L) = 0$. $\blacksquare$

**Proof** of Proposition 5.20. Obviously $W$ is defined where the orientation-preserving condition (4.4) is fulfilled and it approaches infinity as $(\nu, \eta, \mu)$ approaches the boundary of the domain of definition. Set

$$W_0(\nu, \eta, \mu) := \nu^p + |\eta|^p + |\mu|^p, \quad W_1(\nu, \eta, \mu, s) := \frac{1}{\nu^q} + \frac{1}{\nu^2 - h(s)^2 \mu^2} + (q + 2 - p_1) \nu.$$

Clearly $W_0$ is strictly convex and $W_1$ is convex in $(\nu, \eta, \mu)$. Thus $W$ is strictly convex. A simple computation gives that $W(\cdot, \cdot, \cdot, s)$ has a minimum at $(\nu, \eta, \mu) = (1, 0, 0)$. This ensures that the undeformed state is stress free. Hence all constitutive assumptions are fulfilled. Furthermore, $\frac{1}{4} W_0 + W_1$ is bounded below independently of $s$ and we thus readily verify the coercivity condition (5.3). Obviously $E_\lambda(u) \neq +\infty$ for $u = (1, 0, 0, 0)$.

According to (5.8) the trivial solution $u_\lambda^0$ satisfies

$$\lambda = -W_\nu(\nu_\lambda, 0, 0, s) = \frac{1}{\nu_\lambda^{p_1+1}} \left( q + 2 \nu_\lambda^{-2} - p_1 \nu_\lambda^{p_1+q} - (q + 2 - p_1) \nu_\lambda^{p_1+1} \right)$$  \hspace{1cm} (5.61)
We observe that \( \nu_\lambda \) is independent of \( s \) and that

\[
\nu_\lambda \to 0 \quad \text{as} \quad \lambda \to \infty.
\]

Let us choose any subinterval \([s_a, s_b] \subset [0, L]\) and any \( \varepsilon > 0 \) with \( \max_{s_a \leq s \leq s_b} h(s) < \frac{1}{\varepsilon} \). We set

\[
\sigma := \varepsilon \max_{s_a \leq s \leq s_b} h(s), \quad \tilde{W}(\nu, \mu, s) := \frac{1}{\nu^2 - h(s)^2 \mu^2}.
\]

A simple computation yields

\[
\tilde{W}(\nu_\lambda, \varepsilon \nu_\lambda, s) - \tilde{W}(\nu_\lambda, 0, s) = \frac{h(s)^2 \varepsilon^2}{\nu_\lambda^2 (1 - h(s)^2 \varepsilon^2)} \leq \frac{\sigma^2}{\nu_\lambda^2 (1 - \sigma^2)}.
\]

Thus with \( \tilde{\sigma} := \frac{\sigma^2}{1 - \sigma^2} > 0 \)

\[
W(\nu_\lambda, \varepsilon \nu_\lambda, \varepsilon \nu_\lambda, s) - W(\nu_\lambda, 0, 0, s) \leq |\varepsilon \nu_\lambda|^p + |\varepsilon \nu_\lambda|^p + \tilde{\sigma} \leq \frac{1}{\nu_\lambda^2} (\tilde{\sigma} + |\varepsilon \nu_\lambda|^{p+2} + |\varepsilon \nu_\lambda|^{p+2}).
\]

Using (5.61) we obtain that

\[
W(\nu_\lambda, \varepsilon \nu_\lambda, \varepsilon \nu_\lambda, s) - W(\nu_\lambda, 0, 0, s) \leq \frac{1}{\nu_\lambda^2} \tilde{\sigma} + |\varepsilon \nu_\lambda|^{p+2} + |\varepsilon \nu_\lambda|^{p+2} \leq \frac{\tilde{\sigma}}{\nu_\lambda^2}.
\]

Using (5.62) we obtain that

\[
W(\nu_\lambda, \varepsilon \nu_\lambda, \varepsilon \nu_\lambda, s) - W(\nu_\lambda, 0, 0, s) \leq \frac{\tilde{\sigma}}{\nu_\lambda^2} + |\varepsilon \nu_\lambda|^{p+2} + |\varepsilon \nu_\lambda|^{p+2} \leq \frac{\tilde{\sigma}}{\nu_\lambda^2} + \frac{\tilde{\sigma}}{\nu_\lambda^2}.
\]

We immediately verify assertions (a) and (b). In the case of assertion (c) we readily get that

\[
\tilde{W}(\nu_\lambda, \varepsilon \nu_\lambda, s) - \tilde{W}(\nu_\lambda, 0, s) \geq \frac{h^2 \varepsilon^2}{\nu_\lambda^2}
\]

which implies a lower estimate analogously to (5.63). This verifies the assertion first for \( q \geq 2 \), but also for \( q < 2 \).

Proof of Corollary 5.21. If we modify (5.62) and (5.63) for the new \( W \), then we see that the limit does not change.

6 Buckling of rods subjected to obstacles

6.1 Formulation of the problem and results

In this section we investigate the buckling of rods where the deformation is restricted by obstacles. According to our planar setting let us assume that an obstacle \( \mathcal{O} \) is a closed set in the \( \{i, j\}\)-plane with nonempty complement \( \mathcal{O}^c \) and we suppose that the rod can move within the closure \( \text{cl} \mathcal{O}^c \).

Since the handling of lower-dimensional obstacles (e.g., where \( \mathcal{O} \) is a line or a point) is more difficult and not necessary for our purposes, we restrict our attention to obstacles that are the closure of an open set. For our analysis it is convenient to describe the obstacle by an inequality condition

\[
g_0(\mathbf{u}) := \max_{(s, \zeta) \in \Omega} d(\mathbf{p}(\mathbf{u})(s, \zeta)) \leq 0
\]

where

\[
d(\mathbf{q}) := \text{dist}_{\mathcal{O}^c} \mathbf{q} - \text{dist}_{\mathcal{O}} \mathbf{q} \quad \text{for} \quad \mathbf{q} \in \mathbb{R}^2.
\]

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(cf. Schuricht [42]). Observe that \( g_0 \) is well defined. As in the previous section we assume that \( r(0) = 0 \),

\[
g_1(u) = r[u](L) \cdot j = 0,
\]

and we apply a terminal load \(-\lambda i\) at the point \( r(L) \). Then we are looking for critical points of \( E_\lambda \) in \( X \) with respect to the constraint

\[
K = \{ u \in X : g_0(u) \leq 0, g_1(u) = 0 \}
\]
or, equivalently, we are interested in critical points of \((E_\lambda + I_K)\) in \( X \).

We suppose the same general assumptions for the energy density \( W \) as in the previous section stated around (5.3).

Transversality plays an important role for our analysis. Roughly speaking we have to ensure that the reactions caused by the constraints are linearly independent and regular in a certain sense. For this purpose we demand that:

(H1) The boundary \( \partial \Omega \) is Lipschitz, i.e., \( \partial \Omega \) is locally the graph of a Lipschitz function (cf. Evans & Gariepy [25, p. 127]). Furthermore the thickness function \( h : [0, L] \rightarrow [0, \infty) \) is continuous and set \( h_0 := \max \{ h(s) : s \in [0, L] \} \).

(H2) We have that \( \alpha i + \beta j \in \text{cl } \partial \Omega \) for all \( \alpha \in \mathbb{R}, |\beta| \leq h_0 \).

(H3) All end points \( p(0, \zeta) \) and \( p(L, \zeta) \) cannot touch the obstacle for configurations satisfying the constraints.

(H4) If \( u \in K \) and \( E_\lambda(u) < \infty \), then for every \( s \in [0, L] \) the equation \( d(p[u](s, \zeta)) = 0 \) can hold either for \( \zeta = h(s) \) or for \( \zeta = -h(s) \) but not for both.

Since we claim to study buckling problems and not general obstacle problems, condition (H2) is very natural and ensures that the trivial states and, in particular, the trivial solutions \( u_\lambda^0, u_\lambda^{\pm} \) of Theorem 5.6 respect the obstacle. Hypotheses (H3) and (H4) are essential for transversality, where (H4) means that each cross-section can touch the obstacle only with one side. It is satisfied if the obstacle consists of two components and, roughly speaking, the boundary is not curved too much. Observe that contact with the obstacle is only possible for \((s, \zeta) \in \partial \Omega\), since \( p : \Omega \rightarrow \mathbb{R}^2 \) is an open mapping on \( \text{int } \Omega \) for every configuration with finite stored energy (cf. Schuricht [42]).

In the case of an obstacle symmetric with respect to the \( i \)-axis the above hypotheses obviously imply the following stronger version of (H2)

(H2') We have that \( \alpha i + \beta j \notin \partial \Omega \) for all \( \alpha \in \mathbb{R}, |\beta| \leq h_0 \).

As we will see in Proposition 6.8 below, we do not have transversality in the following cases:

(H') One of the points \( p[u](0, h(0)), p[u](0, -h(0)), p[u](L, h(L)), \) or \( p[u](L, -h(L)) \) can touch the obstacle for some admissible \( u \in K \).

(H'') There is some \( \alpha \in \mathbb{R} \) such that \( \alpha i + h_0 j \in \partial \Omega \) and \( \alpha i - h_0 j \in \partial \Omega \) and there is \( s_0 \in ]0, L] \) with \( h(s_0) = h_0 \).
This motivates our general hypotheses. Observe that, by \((H2)\), condition \((H')\) implies that \(h(0) = h_0 \) or \(h(L) = h_0\).

**Theorem 6.1** Let \( E_s \neq +\infty \). The obstacle \( \mathcal{O} \) satisfy hypotheses \((H1) - (H4)\). Then:

(i) For every \( \lambda \in \mathbb{R} \) the trivial solutions \( u_1^\lambda \) and \( u_1^{\pm} \) given by (5.7) and (5.8) are critical points of \( E_{\lambda} + I_K \).

(ii) Suppose that the obstacle \( \mathcal{O} \) is symmetric with respect to the \( i \)-axis and that there is some interval \([s_a, s_b] \subset [0, L]\) and some \( \varepsilon > 0 \) with \( \max_{s_a \leq s \leq s_b} h(s) < \frac{1}{\varepsilon} \) such that

\[
\frac{\int_{s_a}^{s_b} \left( W(\nu(s), \varepsilon \nu(s), \varepsilon \nu(s), s) - W(\nu(s), 0, 0, 0) \right) \, ds}{\lambda \left( \int_{s_a}^{s_b} \nu(s) \, ds \right)^2} \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \infty.
\]

Then for every \( k \in \mathbb{N} \) there is \( \lambda_k > 0 \) such that for all \( \lambda > \lambda_k \) the function \( E_{\lambda} + I_K \) has at least \( k \) distinct pairs \( (u_1, \Phi(u_1)), \ldots, (u_k, \Phi(u_k)) \) of critical points with the property that \( E_{\lambda}(u_j) < \inf \{ E_{\lambda}(u) | u \in \text{Fix}(X) \} \) for \( j = 1, \ldots, k \).

(iii) If there exists a continuous map \( \psi : S^{n-1} \rightarrow \mathcal{D}(E_\lambda) \cap K \) \((S^{n-1} = \text{unit sphere in } \mathbb{R}^n)\) satisfying (2.7), then \( c \) defined in (2.8) is a critical value of \( E_\lambda + I_K \).

**Remark 6.2** (1) In analogy to Remark 5.13.5 we can also here replace the constraint \( r(L) \cdot j = 0 \) by, e.g., \( \theta(0) = 0 \) or \( \theta(L) = 0 \) and we get analogous results. In the last case, however, we have to weaken hypotheses \((H3)\) and to allow that the rod can touch the obstacle also at a point \( p(L, \zeta) \).

(2) Assertion (iii) provides more a strategy for the determination of critical points than a general existence statement for the nonsymmetric case. It is very hard, in general, to determine other critical points than the trivial solutions and the global minimizer of the energy \( E_{\lambda} \) with respect to \( K \). A very detailed study of the special obstacle in relation with the level sets of the energy \( E_{\lambda} \) is necessary to find suitable mappings \( \psi \). For example if \( E_{\lambda} \) has two distinct strong local minimizers with respect to \( K \), then by assertion (iii) we can verify a third critical point which, however, could be some trivial solution. But we do not claim to study this question in more detail. In Schuuricht [41] the construction of some special maps similar to \( \psi \) can be found for a much simpler rod model and with a special obstacle.

We again want to show that the determined critical points satisfy the Euler-Lagrange equations which then justifies our abstract approach also for the case with obstacles.

**Theorem 6.3** Let \( \mathbf{u} \) be any critical point determined in Theorem 6.1. Then there exist \( \lambda_0 \geq 0 \), \( \lambda_1 \in \mathbb{R} \), measurable functions \( s \to N(s), H(s), M(s) \) satisfying (5.15), a probability measure \( \rho \) on \( \Omega \) supported on \( \Omega_{\mathbf{u}} := \{(s, \zeta) \in \Omega : d(\mathbf{p}[\mathbf{u}](s, \zeta)) = 0 \} \) and a \( \rho \)-integrable map \( (s, \zeta) \mapsto d^*(s, \zeta) \in \partial d(\mathbf{p}[\mathbf{u}](s, \zeta)) \) (\( \partial \) with respect to \( d(\cdot) \)) such that \( \lambda_0 \gamma_0(\mathbf{u}) = 0 \) and, with the notation of (5.17) and (6.14),

\[
\mathbf{n}[\mathbf{u}](s) = -\lambda \mathbf{i} - \lambda_1 \mathbf{j} - \tilde{f}_c(s) \quad \text{a.e. on } [0, L],
\]

\[
\mathbf{m}[\mathbf{u}](s) = \lambda \mathbf{r}(s) \times \mathbf{i} - \lambda_1 (\mathbf{r}(L) - \mathbf{r}(s)) \times \mathbf{j}
\]

\[
-\lambda_0 \int_0^L (\mathbf{r}[\mathbf{u}](\tau) \times \tilde{f}_c(\tau)) \, d\tau - \lambda_0 \tilde{f}_c(s) \quad \text{a.e. on } [0, L],
\]

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\[ 0 = -\lambda r(L) \times j - \lambda_0 \int_0^L r'(\tau) \times f_c(\tau) d\tau - \lambda_0 f_{c_0}(0). \quad (6.6) \]

**Remark 6.7**

(1) As in the free case we see that (6.4) and (6.5) coincide with the equilibrium conditions (4.10), (4.11) of our rod theory. However it is very natural that the Euler-Lagrange equations hold only a.e. on \([0, L]\). While in the free case from the previous section this fact is negligible, it becomes important when concentrated forces and, consequently, discontinuities in the strains occur. For a more comprehensive discussion of this fact see Schuricht [42]. Equation (6.6) expresses that \(m(0) = 0\) (cf. (4.7)).

(2) The measure \(\rho\) and the mapping \(d^*\) describe the contact forces exerted by the obstacle. \(\lambda_0\) gives the distribution of this force, \(d^*\) gives the corresponding direction which is normal (in the sense of Clarke [19]) to the boundary of the obstacle. \(\hat{f}_c\) is the distribution of the contact force with respect to the length parameter \(s\) and \(\hat{f}_{c_0}\) is the distribution of the corresponding induced couple (cf. (4.8), (4.9)).

Let us still demonstrate for the presented problem that, in contrast to Lemma 6.17 below, we do not have transversality in the cases where our hypotheses are weakened by \((H')\) or \((H'')\) above. Consequently we cannot apply Theorem 3.4 and in fact also not Theorem 3.13, which we do not have transversality in the cases where our hypotheses are weakened by \((H')\) or \((H'')\).

**Proposition 6.8** Let the obstacle \(O\) satisfy hypotheses \((H1) - (H4)\) weakened by \((H')\) or \((H'')\) and let \((\nu, \eta, \mu, s) \rightarrow W(\nu, \eta, \mu, s)\) be upper-semicontinuous. Then for every \(\lambda \in \mathbb{R}\) there are \(u \in D(E_\lambda)\) with \(g_0(u) \leq 0\) and \(g_1(u) = 0\) such that \((D(E_\lambda), g_0, g_1)\) is not transversal at \(u\).

### 6.2 Proofs

As preparation for the proof of Theorem 6.1 let us verify the following lemmata. Here \(\partial\) denotes the generalized gradient in the sense of Clarke (cf. [19]) and \(\mathbf{C}(\Omega)\) stands for the space of vector-valued continuous functions mapping \(\Omega\) into \(\mathbb{R}^2\).

**Lemma 6.9**

(i) The operator \(A : Y \mapsto \mathbf{C}(\Omega)\) defined by

\[
A(u)(s, \zeta) := p[u](s, \zeta) = \int_0^s \left( \nu(\tau) \mathbf{a}(\theta[\mu, \theta_0](\tau)) + \eta(\tau) \mathbf{b}(\theta[\mu, \theta_0](\tau)) \right) d\tau + \zeta \mathbf{b}(\theta[\mu, \theta_0](s)) \quad (6.10)
\]

with \(\theta[\mu, \theta_0](s)\) according to (4.6) is sequentially continuous from the weak to the strong topology and has a derivative \(A'\) which is uniformly continuous on bounded sets. \(A'\) is given by

\[
\left( A'(u) \hat{u} \right)(s, \zeta) = \int_0^s \left[ \hat{\nu}(\tau) \mathbf{a}(\theta(\tau)) + \hat{\eta}(\tau) \mathbf{b}(\theta(\tau)) + r^{\perp}(\tau) \left( 2 \hat{\theta}_0 + \int_0^\tau \hat{\mu}(\omega) d\omega \right) \right] d\tau - \zeta \mathbf{a}(\theta(s)) \left( 2 \hat{\theta}_0 + \int_0^s \hat{\mu}(\tau) d\tau \right) \quad \text{for } \hat{u} \in Y, \quad (s, \zeta) \in \Omega, \quad (6.11)
\]

where \(\theta(\cdot)\) stands for \(\theta[\mu, \theta_0](\cdot)\) and \(r^{\perp}(\cdot) := \nu(\cdot) \mathbf{b}(\theta(\cdot)) - \eta(\cdot) \mathbf{a}(\theta(\cdot))\).
(ii) The functional $\gamma : C(\Omega) \to \mathbb{R}$ defined by

$$
\gamma(q) := \max_{(s, \zeta) \in \Omega} d(q(s, \zeta)).
$$

is Lipschitz continuous and for every $\gamma^* \in \partial \gamma(q)$, $q \in C(\Omega)$, there exist a probability Borel measure $\rho$ on $\Omega$ supported on $\Omega_\rho := \{(s, \zeta) \in \Omega : \gamma(q) = d(q(s, \zeta))\}$ and a $p$-integrable map $(s, \zeta) \mapsto d^*(s, \zeta) \in \partial d(q(s, \zeta))$ ($\partial$ with respect to $d(\cdot)$) such that for $\hat{q} \in C(\Omega)$

$$
\langle \gamma^*, \hat{q} \rangle = \int_{\Omega} d^*(s, \zeta) \cdot \hat{q}(s, \zeta) \, d\rho(s, \zeta).
$$

(6.12)

(iii) The decomposition $g_0 = \gamma \circ A$ holds and $g_0$ is locally Lipschitz continuous. For every $g^* \in \partial g_0(u)$, $u \in Y$, there exist a probability Borel measure $\rho$ on $\Omega$ supported on $\Omega_\rho := \{(s, \zeta) \in \Omega : g_0(u) = d(p[u](s, \zeta))\}$ and a $p$-integrable map $(s, \zeta) \mapsto d^*(s, \zeta) \in \partial d(p[u](s, \zeta))$ ($\partial$ with respect to $d(\cdot)$) such that for $\hat{u} \in Y$

$$
\langle g^*, \hat{u} \rangle = \int_0^L \left( \hat{\nu}(s) a(\theta(s)) + \hat{\eta}(s) b(\theta(s)) \right) \cdot \hat{f}_c(s) \, ds \\
+ \int_0^L \hat{\mu}(s) \left[ \int_s^L (r'(\tau) \times \hat{f}_c(\tau)) \cdot k \, d\tau + \hat{I}_f(s) \cdot k \right] ds \\
+ 2 \hat{\theta}_0 \left[ \int_0^L (r'(\tau) \times \hat{f}_c(\tau)) \cdot k \, d\tau + \hat{I}_f(0) \cdot k \right]
$$

where

$$
\hat{f}_c(s) := \int_{\Omega_s} d^*(\tau, \zeta) \, d\rho(\tau, \zeta), \quad \hat{I}_f(s) := \int_{\Omega_s} \zeta_b(\tau) \times d^*(\tau, \zeta) \, d\rho(\tau, \zeta),
$$

(6.14)

(iv) The generalized gradient $\partial g_0$ is compact.

(v) If the obstacle $\mathcal{O}$ is symmetric with respect to the $i$-axis, then $g_0$ is $\Phi$-invariant for $\Phi$ defined in (5.4).

PROOF. For the identity in (6.10) observe (4.1), (4.2), (4.5), and (4.6). Using arguments as in the proof of Lemma 5.32 we see that $A$ maps $Y$ into the space $C(\Omega)$ of continuous functions and that it is continuous from the weak to the strong topology. A straightforward computation gives the differentiability of $A$ and relation (6.11). Furthermore, by standard arguments, we see that $A'$ is Lipschitz continuous on bounded sets which implies the uniform continuity on bounded sets.

Since $d : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous with constant 1, we readily verify the Lipschitz continuity of $\gamma$ with constant 1. For the other assertion in (ii) we apply Theorem 2.8.2 in Clarke [19], i.e., for $\gamma^* \in \partial \gamma(q)$ there exist a probability Borel measure $\rho$ on $\Omega$ supported on $\Omega_\rho$ and a mapping $(s, \zeta) \mapsto d^*(s, \zeta) \in \partial d(q(s, \zeta))$ such that $d^*(s, \zeta) \cdot \hat{q}(s, \zeta)$ is $\rho$-integrable for all $\hat{q} \in C(\Omega)$ and that formula (6.12) holds. Without giving the exact definition we only notice that $\partial d(q(s, \zeta))$ denotes a special gradient of $d(\cdot)$ taking into account variations of the function $q(\cdot)$ and of the parameters $(s, \zeta)$. Since $q_i \to q$ in $C(\Omega)$ and $(s_i, \zeta_i) \to (s, \zeta)$ in $\Omega$ implies that $q_i(s_i, \zeta_i) \to q(s, \zeta)$ in $\mathbb{R}^2$, the convexity, weak*-compactness and weak*-closedness (in the sense
of Clarke [19, Prop. 2.1.5]) of generalized gradients gives that \( \partial q d(q(s, \zeta)) = \partial d(q(s, \zeta)) (\partial \) with respect to \( d(\cdot) \) on the right hand side, cf. Clarke [19, p. 27, 29, 85]). If we choose \( \hat{q}(s, \zeta) = i \) and \( \hat{q}(s, \zeta) = j \), respectively, on \( \Omega \), then we see the \( \rho \)-integrability of \( d^*(\cdot, \cdot) \) and assertion (ii) is shown.

The decomposition \( g_0 = \gamma \circ A \) is evident. Recall that \( \gamma \) is Lipschitz continuous and \( A \) is continuously differentiable. Thus \( A \) is locally Lipschitz continuous and induces the same property for the functional \( g_0 \).

Let now \( g^* \in \partial g_0(u), \ u \in Y \). By the chain rule for generalized gradients

\[
\gamma^* \in \partial \gamma(A(u)) \circ A'(u)
\]

(cf. Clarke [19, p.45]). Thus there is some \( \gamma^* \in \partial \gamma(A(u)) \) such that for \( \hat{u} \in Y \)

\[
\langle \gamma^*, \hat{u} \rangle = \langle \gamma^*, A'(u) \hat{u} \rangle.
\]

By assertion (ii) and formula (6.11) we can find a measure \( \rho \) and a measurable mapping \( d^* \) with the properties described in assertion (iii) such that

\[
\langle g^*, \hat{u} \rangle = \int_{\Omega} d^*(s, \zeta) \cdot \left[ \int_0^s \left( \hat{\mu}(\tau) a(\theta(\tau)) + \hat{\eta}(\tau) b(\theta(\tau)) \right) d\tau \right] d\rho(s, \zeta)
\]

\[
+ \int_{\Omega} d^*(s, \zeta) \cdot \left[ \int_0^s r^\perp(\tau) \left( 2 \hat{\theta}(\tau) + \int_0^\tau \hat{\mu}(\omega) d\omega \right) d\tau \right] d\rho(s, \zeta)
\]

\[
- \int_{\Omega} d^*(s, \zeta) \cdot \left[ \zeta a(\theta(s)) \left( 2 \hat{\theta}(\tau) + \int_0^s \hat{\mu}(\tau) d\tau \right) \right] d\rho(s, \zeta).
\]

Applying Fubini’s Theorem we obtain formula (6.13) (cf. also Schuricht [42, Section 6.2]). This verifies assertion (iii).

Assertion (iv) is a consequence of Proposition 3.14.

Let us now prove (v). Recalling (4.1), (4.2), (4.5), (4.6) we readily see that

\[
p[\Phi(u)](s, -\zeta) \cdot i = p[u](s, \zeta) \cdot i, \quad p[\Phi(u)](s, -\zeta) \cdot j = -p[u](s, \zeta) \cdot j.
\]

Let \( \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \) denote the reflection at the \( i \)-axis. Then (6.16) means that

\[
p[\Phi(u)](s, -\zeta) = \Psi(p[u](s, \zeta)).
\]

The symmetry of the obstacle gives that \( d(q) = d(\Psi(q)) \) for \( q \in \mathbb{R}^2 \). Since \( \Omega \) is symmetric with respect to \( \zeta = 0 \), we finally obtain

\[
g_0(\Phi(u)) = \max_{(s, \zeta) \in \Omega} d(p[\Phi(u)](s, \zeta)) = \max_{(s, \zeta) \in \Omega} d(p[\Phi(u)](s, -\zeta))
\]

\[
= \max_{(s, \zeta) \in \Omega} d(\Psi(p[u](s, \zeta))) = \max_{(s, \zeta) \in \Omega} d(p[u](s, \zeta)) = g_0(u).
\]

\[\rule{1cm}{0.1mm}\]

**Lemma 6.17** Let the obstacle satisfy (H1) – (H4). Then:

(i) \( (D(E_\lambda), g_0, g_1) \) is transversal at any \( u \in D(E_\lambda) \subset Y \) with \( g_0(u) \leq 0 \) and \( g_1(u) = 0 \).

(ii) \( (E_\lambda, g_0, g_1) \) is uniformly transversal at any level \( c \in \mathbb{R} \).
PROOF. Let us start with assertion (i). We choose \( u \in D(E_\lambda) \) with \( g_0(u) \leq 0 \) and \( g_1(u) = 0 \). If \( g_0(u) < 0 \), then we can argue as in the proof of Lemma 5.41. We suppose now that \( g_0(u) = 0 \). Define

\[
\mathcal{I}_k := \{ s \in [0, L] : W(\tilde{\nu}, \tilde{\eta}, \mu(s), s) \leq k \text{ for all } |\tilde{\nu} - \nu(s)| \leq \frac{1}{k}, |\tilde{\eta} - \eta(s)| \leq \frac{1}{k} \}. \tag{6.18}
\]

The sets \( \mathcal{I}_k \) are measurable and \([0, L] \setminus \bigcup_{k=1}^{\infty} \mathcal{I}_k \) has measure zero by \( u \in D(E_s) \).

By (H3) there exist \( s_0, s_L \in ]0, L[ \) such that

\[
d(p[u](s, \zeta)) < 0 \quad \text{for all } (s, \zeta) \in \Omega \text{ with } s \in [0, s_0] \cup [s_L, L].
\]

For some large \( k_0 \in \mathbb{N} \) we have that \( \mathcal{I}_L := \mathcal{I}_0 \cap [s_L, L] \) has nonzero measure. We set

\[
u_L(s) = \begin{cases} \frac{1}{k_0} \sin \theta(s) & \text{for } s \in \mathcal{I}_L, \\ 0 & \text{otherwise}, \end{cases} \quad \eta_L(s) = \begin{cases} \frac{1}{k_0} \cos \theta(s) & \text{for } s \in \mathcal{I}_L, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( \theta(s) := \theta(u)(s) \). Obviously \( u_L := (\nu_L, \eta_L, 0, 0) \in Y \) and by (5.34) we have

\[
\langle g'_1(u), u_L \rangle = \frac{1}{k_0} \int_{\mathcal{I}_L} \sin^2 \theta + \cos^2 \theta \, ds > 0. \tag{6.19}
\]

We now construct some special variations transversal to the obstacle at \( u \). By the growth condition (4.20) and by \( E_s(u) < \infty \) it is proven in Schuricht [42] that \( p[u](\cdot, \cdot) \) is an open mapping on \( \text{int}(\Omega) \). Because \( h(\cdot) \) is continuous, \( d(p[u](s, \zeta)) = 0 \) implies that \( \zeta = \pm h(s) \), i.e., contact can only occur for cross-sections corresponding to one of the sets

\[
\mathcal{I}_c^+ := \{ s \in [0, L] : d(p[u](s, h(s))) = 0 \}, \quad \mathcal{I}_c^- := \{ s \in [0, L] : d(p[u](s, -h(s))) = 0 \}.
\]

By the continuity of \( d(\cdot) \) and \( p[u](\cdot, \cdot) \) and by (H4) the sets \( \mathcal{I}_c^\pm \) are compact and there exist open neighborhoods \( U^\pm \supset \mathcal{I}_c^\pm \) with \( U^+ \cap U^- = \emptyset \) and \( U^+ \cup U^- \subset [s_0, s_L] \). Since the boundary \( \partial \mathcal{O} \) is Lipschitz, \( 0 \notin \partial d(q) \) if \( d(q) = 0 \). By the upper semicontinuity of \( \partial d(\cdot) \) we thus can find finitely many open intervals \( \{ \mathcal{I}_{c,j}^+ \}_{j=1}^{m_1} \) and \( \{ \mathcal{I}_{c,j}^- \}_{j=m_1+1}^{m_2} \) which are subsets of \( U^+ \) and \( U^- \), respectively, and which cover \( \mathcal{I}_c^+ \) and \( \mathcal{I}_c^- \), respectively, and there exist unit vectors \( w_j \in \mathbb{R}^2, j = 1, \ldots, m_2 \), such that

\[
w_j \cdot d^* < 0 \quad \text{for all } d^* \in \partial d(p[u](s, \pm h(s))), \quad s \in \mathcal{I}_c^j, \tag{6.20}
\]

where “\( \pm \)” is to choose according to \( \mathcal{I}_{c,j}^+ \subset U^\pm \). Let us denote the left end, the center, and the right end of the interval \( \mathcal{I}_{c,j}^+ \) by \( s_{j}^l, s_j, \) and \( s_{j}^r \), respectively. For some possibly larger \( k_0 \) there is \( \delta > 0 \) such that

\[
\delta^l_j := \text{meas} \left( [s_{j}^l, s_j \cap \mathcal{I}_{k_0}] \right) > \delta, \quad \delta^r_j := \text{meas} \left( [s_j, s_{j}^r \cap \mathcal{I}_{k_0}] \right) > \delta, \quad j = 1, \ldots, m_2.
\]

We now define

\[
t_j(s) := \int_0^s t_j'(\tau) \, d\tau \quad \text{with} \quad t_j'(s) = \begin{cases} \frac{\delta}{k_0 m_2 \delta^l_j} & \text{on } [s_{j}^l, s_j \cap \mathcal{I}_{k_0}], \\ -\frac{\delta}{k_0 m_2 \delta^r_j} & \text{on } [s_j, s_{j}^r \cap \mathcal{I}_{k_0}], \\ 0 & \text{otherwise}.
\end{cases}
\]

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We readily see that \( k_0 \) can be chosen so large that for every \( s \in \mathcal{I}_c^+ \cup \mathcal{I}_c^- \) there is some index \( j \) with \( t_j(s) > 0 \). Observe that

\[
0 \leq t_j(s) \leq \frac{\delta}{k_0 m_2}, \quad t_j(s) = 0 \text{ for } s \notin \mathcal{I}_c^j, \quad |t'_j(s)| < \frac{1}{k_0 m_2}.
\]  

(6.21)

We introduce \( \hat{u} := (\hat{\nu}, \hat{\eta}, 0, 0) \in Y \) with

\[
\hat{\nu}(s) := \sum_{j=1}^{m_2} t'_j(s) a(s) \cdot w_j, \quad \hat{\eta}(s) := \sum_{j=1}^{m_2} t'_j(s) b(s) \cdot w_j,
\]

and

\[
u_{\pm} := u + \hat{\nu}_{\pm} \quad \text{where} \quad \hat{\nu}_{\pm} := \hat{\nu} \pm u_L.
\]

(6.22)

Recalling (6.18) and (6.21) we see that \( u_+, u_- \in D(E_\lambda) \). Choose now any \( g^* \in \partial g_0(u) \). By Lemma 6.9 the gradient \( g^* \) corresponds to a probability measure \( \rho \), supported on \( \Omega_u \), and a \( \rho \)-integrable map \((s, \zeta) \mapsto \hat{d}^*(s, \zeta) \in \partial d(\hat{p}[u](s, \zeta)) \). Since the set \( \{(s, \zeta) \in \Omega : s \in [s_L, L]\} \) does not belong to the support of \( \rho \), we obtain by (6.15) and (6.20) that

\[
\langle g^*, \hat{\nu}_{\pm} \rangle = \langle g^*, \hat{u} \rangle = \int_{\Omega} \hat{d}^*(s, \zeta) \cdot \int_0^s \left( \hat{\nu}(\tau) a(\tau) + \hat{\eta}(\tau) b(\tau) \right) d\tau d\rho(s, \zeta)
\]

\[
= \sum_{j=1}^{m_2} \int_{\Omega} t_j(s) w_j \cdot \hat{d}^*(s, \zeta) d\rho(s, \zeta) < 0.
\]

(6.23)

Recall that the generalized directional derivative can be determined by

\[
g_0^0(u; v) = \max_{g^* \in \partial g_0(u)} \langle g^*, v \rangle = \langle \hat{g}^*, v \rangle
\]

for some \( \hat{g}^* \in \partial g_0(u) \) (cf. Clarke [19]). Since in (6.23) the gradient \( g^* \in \partial g_0(u) \) was arbitrary, we get

\[
g_0^0(u; u_+ - u) = g_0^0(u; \hat{\nu}_+) < 0, \quad g_0^0(u; u_- - u) = g_0^0(u; \hat{\nu}_-) < 0.
\]

Using (5.34) and (4.2) we obtain

\[
\langle g'_1(u), \hat{u} \rangle = \int_0^L \left( \hat{\nu}(s) a(s) \cdot j + \hat{\eta}(s) b(s) \cdot j \right) ds
\]

\[
= \int_0^L \sum_{j=1}^{m_2} t'_j(s) w_j ds = \int_0^L \sum_{j=1}^{m_2} t'_j(L) w_j = 0.
\]

Consequently by (6.19)

\[
\pm \langle g'_1(u), u_{\pm} - u \rangle = \langle g'_1(u), u_L \rangle > 0.
\]

But this verifies the transversality at \( u \) stated in assertion (i).

Let now \( (u_n) \) be a bounded sequence in \( K \) with \( E_\lambda(u_n) \to c \) for some \( c \in \mathbb{R} \). There exist a subsequence, denoted the same way, that weakly converges to some \( u \in Y \). As in the proof of Lemma 5.41 we get \( u \in D(E_\lambda) \) and \( g_1(u) = 0 \). If \( \liminf_{n \to \infty} g_0(u_n) < 0 \), then we can proceed as in the proof of Lemma 5.41. Let us now assume that \( \lim_{n \to \infty} g_0(u_n) = 0 \). According to \( u \) we

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construct \( u_+ \), \( u_- \) as in the proof of assertion (i) (cf. (6.22)). By Lemma 6.9 and Proposition 3.14 we get
\[
\limsup_{n \to \infty} g_0^0(u_n; u_- - u_n) \leq \limsup_{n \to \infty} \gamma_0^0(A(u_n); A'(u_n)(u_- - u_n)) \\
\leq \gamma_0^0(A(u); A'(u)(u_- - u)). \tag{6.25}
\]
As in (6.24) we can find \( \gamma_+^*, \gamma_-^* \in \partial \gamma(A(u)) \) with
\[
\gamma_0^0(A(u); A'(u)(u_- - u)) = \langle \gamma_\pm^*, A'(u)(u_- - u) \rangle. \tag{6.26}
\]
By Lemma 6.9 there exist probability Borel measures \( \rho_\pm \) on \( \Omega \) supported on \( \Omega_{A(u)} = \Omega_u \) and \( \rho \)-integrable mappings \((s, \zeta) \mapsto d_\rho^\pm(s, \zeta) \in \partial d(A(u)(s, \zeta)) = \partial d(p[u](s, \zeta)) \) such that
\[
\langle \gamma_\pm^*, A'(u)(u_- - u) \rangle = \int_{\Omega} d_\rho^\pm(s, \zeta) \cdot [A'(u)(u_- - u)](s, \zeta) \ d\rho_\pm(s, \zeta).
\]
Using (6.11) we get, in analogy to (6.23),
\[
\langle \gamma_\pm^*, A'(u)(u_- - u) \rangle = \langle \gamma_\pm^*, A'(u)\hat{u} \rangle = \int_{\Omega} d_\rho^\pm(s, \zeta) \cdot [A'(u)\hat{u}](s, \zeta) \ d\rho_\pm(s, \zeta) \\
< \int_{\Omega} d_\rho^\pm(s, \zeta) \cdot \left[ \int_0^t \left( \tilde{\nu}(\tau) a(\theta(\tau)) + \tilde{\eta}(\tau) b(\theta(\tau)) \right) \ d\tau \right] \ d\rho_\pm(s, \zeta) = 0.
\]
With (6.25) and (6.26) this implies assertion (ii). □

Proof of Theorem 6.1. We argue analogously as in the proof of Theorem 5.6. For assertion (i) observe that \( g_0(u_\lambda^0) \leq 0 \), that \( K \) is closed and apply Proposition 3.1. In the case of assertion (ii) we use, in addition, the following arguments. By the symmetry of the obstacle \( \mathcal{O} \) condition (H2') is met and hence \( g_0(u_\lambda^0) < 0 \). Furthermore \( K \) is \( \Phi \)-invariant by Lemma 6.9. For transversality we use Lemma 6.17 instead of Lemma 5.41. The compactness of \( \partial g_0 \) is stated in Lemma 6.9. Finally we have to ensure that all \( \psi_k^\lambda(x) \) belong to \( K \). Since \( g_0 \) is continuous, \( g_0(u_\lambda^0) < 0 \), and all \( \psi_k^\lambda(x) \) are small variations of \( u_\lambda^0 \), we can choose \( \ell \) so small in (5.58) that \( \psi_k^\lambda(x) \in K \) for all \( x \in S^{k-1} \). Assertion (iii) directly follows from Theorem 2.6. □

Proof of Theorem 6.3. We argue analogously as in the proof of Theorem 5.14. Transversality follows by Lemma 6.17 for this case and the generalized gradient \( \partial g_0(u) \) is described in Lemma 6.9. This way we readily obtain the assertion. □

Proof of Proposition 6.8.

(a) We start with the case (H') where \( p[\hat{u}](0, h(0)) \in \mathcal{O} \) or \( p[\hat{u}](0, -h(0)) \in \mathcal{O} \) for some \( \hat{u} \in K \). By (H2) this can only hold if \( h(0) = h_0 \) and \( h_0j \in \mathcal{O} \) or \( -h_0j \in \mathcal{O} \). Without loss of generality we can restrict our attention to the case where \( p[\hat{u}](0, h(0)) = h_0j \in \mathcal{O} \). For some constant function \( \nu > 0 \) we consider the trivial state \( u := (\nu, 0, 0, 0) \in Y \) which can differ from \( \hat{u} \). Clearly \( u \in K \). By the upper-semicontinuity of \( W \) the function \( W(\nu, 0, 0, \cdot) \) is bounded above on \([0, L] \) and, therefore, \( u \in D(E_\lambda) \) for any \( \lambda \in \mathbb{R} \). We will show that
\[
g_0^0(u; \hat{u}) \geq 0 \quad \text{for all } \hat{u} \in Y, \tag{6.27}
\]
which prevents transversality of \((D(E_\lambda), g_0, g_1)\) at \(u\).

Clearly \(g_0(u) = 0\). Let us first choose \(\hat{u} \in Y\) with \(\hat{\theta}_0 = 0\). Then \(g_0(u + t \, \hat{u}) \geq 0\) for all \(t \in \mathbb{R}\). Consequently

\[
g_0^0(u; \hat{u}) = \limsup_{t \downarrow 0} \frac{g_0(u + t \, \hat{u}) - g_0(u)}{t} \geq 0.
\]

(6.28)

Assume now that \(\hat{\theta}_0 \neq 0\). For \(n \in \mathbb{N}\) we define \(u_n \in Y\) and \(t_n\) according to

\[
\theta_{0,n} := -\frac{1}{n} \hat{\theta}_0, \quad \nu_n := \nu \cos \theta_{0,n}, \quad \eta_n := -\nu \sin \theta_{0,n}, \quad \mu_n := 0, \quad t_n := \frac{1}{n}.
\]

Clearly \(u_n \to u\). Recalling (4.1) we see that \(p[u_n](s, \zeta) = \nu s i + \zeta (-\sin \theta_{0,n} i + \cos \theta_{0,n} j)\). Hence \(|j \cdot p[u_n](s, \zeta)| < h_0\) and \(g_0(u_n) < 0\). Observing \(\theta_{0,n} + t_n \hat{\theta}_0 = 0\), we get \(p[u_n + t_n \hat{u}](0, h_0) = h_0 j \in O\). Therefore \(g_0(u_n + t_n \hat{u}) \geq 0\) and

\[
g_0^0(u; \hat{u}) = \limsup_{n \to \infty} \frac{g_0(u_n + t_n \hat{u}) - g_0(u_n)}{t_n} \geq 0.
\]

(6.29)

(b) We now consider the case \((H')\) where \(p[\hat{u}](L, h(L)) \in O\) or \(p[\hat{u}](L, -h(L)) \in O\) for some \(\hat{u} \in K\). We must have that \(h(L) = h_0\) and, without loss of generality, we can suppose that \(p[\hat{u}](L, h(L)) = \alpha i + h_0 j \in O\) for some \(\alpha \geq 0\).

(b1) First let us consider the case where \(\alpha > 0\). We study the trivial state \(u = (\nu, 0, 0, 0) \in Y\) with \(\nu(s) = \alpha /L\). Obviously \(g_1(u) = 0\), \(p[u](L, h_0) = \alpha i + h_0 j =: \hat{q}\), \(g_0(u) = 0\), and \(E_\lambda(u)\) is finite.

The structure of \(\partial g_0(u)\) is given in Lemma 6.9. Observe, however, that it can deliver a set bigger than \(\partial g_0(u)\). Let us consider \(g^* \in Y^*\) defined as in (6.13) corresponding to \(\rho\) and \(d^*\) where the support of \(\rho\) is the singleton \(\{(L, h_0)\}\) and \(d^*(L, h_0) = j\). We will show below that really \(g^* \in \partial g_0(u)\). Let us mention that \(j \in \partial d(\hat{q})\), but this is not directly needed for our conclusions. Using (6.13), (4.2), and (5.39) we readily compute for \(\hat{u} \in Y\)

\[
\langle g^*, \hat{u} \rangle = \int_0^L \left( \hat{\mu}(s) a(s) \cdot j + \hat{\eta}(s) b(s) \cdot j \right) ds + k \cdot \int_0^L \left( \hat{\mu}(s) \int_s^L (r'(r) \times j) d\tau \right) ds + 2 \hat{\theta}_0 k \cdot \int_0^L (r'(r) \times j) d\tau
\]

\[
= \int_0^L \left( \hat{\mu} \sin \theta + \hat{\eta} \cos \theta \right) ds + i \cdot \int_0^L \hat{\mu}(s) \left( r(L) - r(s) \right) ds + 2 \hat{\theta}_0 i \cdot r(L)
\]

\[
= \int_0^L \left( \hat{\eta} + \hat{\theta} \nu \right) ds.
\]

Then by (5.34)

\[
\langle g^*, \hat{u} \rangle = \langle g^*_1(u), \hat{u} \rangle \quad \text{for all} \quad \hat{u} \in Y.
\]

(6.30)

According to Definition 3.3 transversality at \(u\) in particular implies that we can find \(\hat{u}_+ \in Y\) with

\[
g_0^0(u; \hat{u}_+) < 0, \quad -g_0^0(u; -\hat{u}_+) = \langle g^*_1(u), \hat{u}_+ \rangle > 0.
\]

(6.31)

Using (6.24) we see that (6.30) contradicts (6.31), i.e., we do not have transversality at \(u\).
Let us now show that \( g^* \in \partial g_0(u) \). We set
\[
\hat{d}(q) := -\|q - \hat{q}\| \quad \text{for } q \in \mathbb{R}^2,
\]
\[
t_n := \frac{h_0}{n^2}, \quad q_n := \frac{h_0}{n} j, \quad n \in \mathbb{N}.
\]
Let us now choose any \( \hat{u} \in Y \) and with \( A \) according to Lemma 6.9 we define
\[
\hat{p} := \left(A'(u) \right) \hat{u}(L, h_0).
\]
Clearly \( \hat{d}(q) = (q - q)/\|q - q\| \) for \( q \neq q \). Thus for \( \hat{q} \in \mathbb{R}^2 \) and \( n \in \mathbb{N} \) sufficiently large there exist \( \tau_n \in (0, 1) \) such that
\[
\hat{d}(q_n + t_n \hat{q}) - \hat{d}(q_n) = \hat{d}(q_n + \tau_n t_n \hat{q}) t_n \hat{q} = \frac{j - \frac{1}{n} \tau_n \hat{q} \hat{q}}{\|j - \frac{1}{n} \tau_n \hat{q}\|}.
\]
Recalling (6.11), (6.15), we see that
\[
\langle g^*, \hat{u} \rangle = j \cdot \hat{p} = \lim_{n \to \infty} \frac{\hat{d}(q_n + t_n \hat{p}) - \hat{d}(q_n)}{t_n}.
\]
Since \( \hat{d}(q_n + t_n \hat{p}) \leq d(q_n + t_n \hat{p}) \) and \( \hat{d}(q_n) = d(q_n) \) for large \( n \in \mathbb{N} \),
\[
\langle g^*, \hat{u} \rangle \leq \limsup_{n \to \infty} \frac{d(q_n + t_n \hat{p}) - d(q_n)}{t_n}. \tag{6.32}
\]
Let us study \( u_n \in Y \) given by
\[
(\nu_n, \eta_n, \mu_n, \theta_{0,n}) = (d_n \sin \theta_n + c_n \cos \theta_n, d_n \cos \theta_n - c_n \sin \theta_n, 0, \theta_{0,n}),
\]
\[
\theta_n(s) = 2\theta_{0,n} \in (-\frac{\pi}{2}, 0) \quad \text{such that} \quad \cos 2\theta_{0,n} = 1 - \frac{2}{n},
\]
\[
c_n = \frac{\alpha + h_0 \sin 2\theta_{0,n}}{L}, \quad d_n(s) = \begin{cases} \frac{2h_0}{L} n \quad \text{on} \quad \left[0, \frac{L}{2}\right], \\ 0 \quad \text{on} \quad \left(\frac{L}{2}, L\right]. \end{cases}
\]
This yields that
\[
p[u_n](s, \zeta) = \begin{cases} \frac{\alpha \sin \theta_n}{L} s \cos \theta_n \cos \zeta \sin \theta_n + \left(\frac{2h_0}{L} s + \zeta(1 - \frac{2}{n})\right) i + \left(\frac{h_0}{n} + \zeta(1 - \frac{2}{n})\right) j & \text{if} \quad s \in [0, \frac{L}{2}], \\ \frac{\alpha \sin \theta_n}{L} s \cos \theta_n \cos \zeta \sin \theta_n + \left(\frac{h_0}{n} + \zeta(1 - \frac{2}{n})\right) i + \left(\frac{2h_0}{L} s + \zeta(1 - \frac{2}{n})\right) j & \text{if} \quad s \in \left(\frac{L}{2}, L\right]. \end{cases}
\]
We readily see that \( u_n \to u \) in \( Y \), \( p[u_n](L, h_0) = q_n \), and for large \( n \in \mathbb{N} \)
\[
\max_{(s, \zeta) \in \Omega} d(p[u_n](s, \zeta)) = d(p[u_n](L, h_0)). \tag{6.33}
\]
Because \( A \) is continuously differentiable we have that
\[
\lim_{n \to \infty} \frac{1}{t_n} \left\| A(u_n + t_n \hat{u}) - A(u_n) - A'(u) t_n \hat{u} \right\| = \lim_{n \to \infty} \left\| \int_0^1 \left( A'(u_n + \tau t_n \hat{u}) - A'(u) \right) d\tau \hat{u} \right\| \leq \lim_{n \to \infty} \sup_{\tau \in [0, 1]} \| A'(u_n + \tau t_n \hat{u}) - A'(u) \| \| \hat{u} \| = 0.
\]
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This implies that
\[ \lim_{n \to \infty} \frac{1}{n} w_n = 0 \text{ for } w_n := p[u_n + t_n \hat{u}](L, h_0) - q_n - t_n \hat{P}. \]

The Lipschitz continuity of \( d \) (with Lipschitz constant 1) then ensures that
\[ d(q_n + t_n \hat{P}) \leq d(p[u_n + t_n \hat{u}](L, h_0)) + \|w_n\|. \]

Inequality (6.32) now yields
\[ \langle g^*, \hat{u} \rangle \leq \limsup_{n \to \infty} \frac{1}{t_n} \left( d(p[u_n + t_n \hat{u}](L, h_0)) - d(p[u_n](L, h_0)) + \|w_n\| \right) \]
\[ \leq \limsup_{n \to \infty} \frac{1}{t_n} \left( \max_{(s, \zeta) \in \Omega} d(p[u_n + t_n \hat{u}](s, \zeta)) - \max_{(s, \zeta) \in \Omega} d(p[u_n](s, \zeta)) \right) \]
\[ = \limsup_{n \to \infty} \frac{1}{t_n} \left( g_0(u_n + t_n \hat{u}) - g_0(u_n) \right) \]
\[ \leq \limsup_{v \to 0} \left( \frac{g_0(v + t \hat{u}) - g_0(v)}{t} \right) = g_0^0(u; \hat{u}). \]

The arbitrariness of \( \hat{u} \in Y \) and the definition of the generalized gradient imply that \( g^* \in \partial g_0(u) \).

(b2) Let us consider the case \( \alpha = 0 \). With \( \hat{s} := \frac{\pi}{6} \) we define \( u \in Y \) by
\[ (\nu, \eta, \mu, \theta_0) = \begin{cases} (c_0 \cos \frac{3\pi}{8}, -c_0 \sin \frac{3\pi}{8}, 0, \frac{3\pi}{16}) & \text{on } [0, \hat{s}], \\ (-\frac{h_0}{2\pi} \sin \frac{3\pi}{8} + c_1 \cos \frac{3\pi}{8}, -\frac{h_0}{2\pi} \cos \frac{3\pi}{8} - c_1 \sin \frac{3\pi}{8}, 0, \frac{3\pi}{16}) & \text{on } [\hat{s}, 2\hat{s}], \\ (\frac{h_0}{2\pi} \sin \theta, \frac{h_0}{2\pi} \cos \theta, \frac{\pi}{8}, \frac{3\pi}{16}) & \text{on } [2\hat{s}, 4\hat{s}], \\ (\frac{h_0}{2\pi} \sin \frac{3\pi}{8} - c_1 \cos \frac{3\pi}{8}, \frac{h_0}{2\pi} \cos \frac{3\pi}{8} + c_1 \sin \frac{3\pi}{8}, 0, \frac{3\pi}{16}) & \text{on } [4\hat{s}, 5\hat{s}], \\ (-c_0 \cos \theta, c_0 \sin \theta, \frac{3\pi}{16}) & \text{on } [5\hat{s}, L]. \end{cases} \]

With the constants \( c_0 = \frac{\pi h_0}{8} \), \( c_1 = \frac{h_0}{4} \tan \frac{3\pi}{8} \) and with \( \theta \) according to (4.6) we verify that (4.4) is satisfied. By the upper-semicontinuity of \( W \) we see that \( u \in D(E_s) \). Some simple computations using (4.5) yield that \( r(u)(s) \) lies on the i-axis for \( s \in [0, \hat{s}] \cup [5\hat{s}, L] \) and \( r(u)(L) = 0 \). On \([\hat{s}, 5\hat{s}]\) the reference line forms a triangle symmetric with respect to the i-axis and with the corners \( r(u)(\hat{s}) = r(u)(5\hat{s}) = c_0 \hat{s} i, r(u)(2\hat{s}) = (c_0 + c_1) \hat{s} i - \frac{h_0}{\pi} j, r(u)(4\hat{s}) = (c_0 + c_1) \hat{s} i + \frac{h_0}{2\pi} j \). For the angels we get \( \theta(0) = \theta(\hat{s}) = \theta(2\hat{s}) = \frac{3\pi}{8}, \theta(3\hat{s}) = \frac{\pi}{8}, \theta(4\hat{s}) = \frac{5\pi}{8}, \theta(L) = \pi \). Thus \( p(u)(L, -h_0) = h_0 \hat{P} \) and there is no contact for \( s < L \), i.e., \( u \in K \). Now we can modify the arguments from (b1) to the part of the rod corresponding to \( s \in [5\hat{s}, L] \) and we obtain the assertion.

(c) Let us finally study the case (\( H^* \)). Without loss of generality we can restrict our attention to \( \alpha \geq 0 \). In analogy to (b) we construct a state \( u \) such that \( p(u)(s_0, \pm h_0) \in \Omega \) and \( (\nu, \eta, \mu, \theta_0) = (1, 0, 0, \theta_0) \) on \([s_0, L] \). Then we study \( g^*_\pm \in Y^* \) corresponding to \( \rho_\pm \) supported at \((s_0, \pm h_0) \) and \( d^*_\pm \) with \( d^*_\pm(s_0, \pm h_0) = \pm j \). We analogously obtain that \( g^*_\pm \in \partial g_0(u) \) and we readily see that \( g^-_\pm = -g^*_\pm \). By (6.24) this yields that \( g^*_0(u; \hat{u}) \geq 0 \) for all \( \hat{u} \in Y \) which contradicts transversality at \( u \).

References


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