Nonlinear Treatment of Eigenvalue Problems

1. Newton’s Method\(^1\) for Simple Characteristic Roots

Let the linear special eigenvalue problem

\[
g = (\mathbb{A} - \lambda \mathbb{E}) x = \mathcal{M}(\lambda) x = 0 \quad \text{with} \quad \mathcal{M}(\lambda) = (m_1, \ldots, m_n) \quad (1)
\]

be given. Wanted: The characteristic root \(\lambda_i\) with the corresponding eigenvector \(x_i = \{x_1, x_2, \ldots, x_n\}\). If one sets \(x_i = 1\) (or, if necessary, another component) then the \(n\) components of the vector \(v = \{\lambda_i, x_2, x_3, \ldots, x_n\}\) are unknown. One can consider (1) as a ‘nonlinear system of equations’ in the components of \(v\) and thus turn to a number of approximation methods for determining the characteristic root and the eigenvector. Here we will focus on Newton’s method\(^2\). Starting from a known approximating vector \(v_0\) one obtains the Newton correction

\[
\delta v_\nu = \{\delta \lambda^{(\nu)}, \delta x_2^{(\nu)}, \ldots, \delta x_n^{(\nu)}\} \quad \text{for} \quad \nu = 1, 2, 3, \ldots \quad \text{by}
\]

\[
g_\nu + \left( \frac{\partial g}{\partial v} \right)_\nu \delta v_\nu = 0 \quad \text{with} \quad \left( \frac{\partial g}{\partial v} \right)_\nu = \mathfrak{G}_\nu = (-\delta \lambda_i^{(\nu)}, m_2^{(\nu)}, \ldots, m_n^{(\nu)}). \quad (2)
\]

In case of a simple characteristic root \(\lambda_i\) with corresponding eigenvector \(x_i\), the determinant of the functional matrix \(\mathfrak{G}_\nu\) is different from zero.

If \(k_l^{(\nu)}\) is the deviation of the \(l\)-th component in \(v_\nu\) from the exact value then there holds

\[
k_l^{(\nu+1)} \leq C |k_l^{(\nu)}| K^{(\nu)} \quad \text{with} \quad |k_l^{(\nu)}| \leq K^{(\nu)} \quad \text{for} \quad l = 1, 2, \ldots, n. \quad (3)
\]

\((C\) is a constant that can be estimated; in case of a linear system there holds \(k_l^{(\nu)} = 0\).\) For \(CK^{(\nu)} < 1\) one has convergence, actually quadratic convergence. An error estimate can also be given.

If the functional matrix in (2) is kept constant, quadratic convergence is lost but the computations become especially simple.

2. Other Applications of Newton’s Method

Newton’s method can also be applied in case of multiple characteristic roots. Furthermore, Newton’s method can be applied in a similar manner to the general linear eigenvalue problem \(\mathcal{M}(\lambda) = \mathbb{A} - \lambda \mathbb{B}\) as well as the nonlinear eigenvalue problem, e.g., \(\mathcal{M}(\lambda) = \mathbb{A}\lambda^2 + \mathbb{B}\lambda + \mathbb{C}\).

\(^1\)A detailed publication about 'Newton’s Method for Eigenvalue Problems' is in preparation where, among others, the connection with the 'inverse iteration' is addressed. I am grateful to Prof. Wielandt for bringing this connection to my attention during the conference.

3. Complex Characteristic Roots for the Special Eigenvalue Problem with Real Matrix

With \( \lambda = \rho + i \sigma \) and \( x = \eta + i \zeta \) one can formulate a \( 2n \)-dimensional real problem

\[
\begin{pmatrix}
\mathbb{A} - \rho \mathbb{E} & \sigma \mathbb{E} \\
-\sigma \mathbb{E} & \mathbb{A} - \rho \mathbb{E}
\end{pmatrix}
\begin{pmatrix}
\eta \\
\zeta
\end{pmatrix} = 0
\]

\[
\begin{align*}
\zeta &= -\frac{1}{\sigma}(\mathbb{A} - \rho \mathbb{E}) \eta \\
\eta &= \frac{1}{\sigma}(\mathbb{A} - \rho \mathbb{E}) \zeta.
\end{align*}
\] (4)

The elimination of real or imaginary part leads to one and the same nonlinear problem

\[
[(\mathbb{A} - \rho \mathbb{E})^2 + \sigma^2] \eta = 0 \quad \text{and} \quad [(\mathbb{A} - \rho \mathbb{E})^2 + \sigma^2] \zeta = 0, \quad \text{resp.}
\] (5)

The nonlinear system (5) is treated with Newton’s method and yields \( \eta \) or \( \zeta \). The missing imaginary part \( \zeta \) and real part \( \eta \), resp., can then be recovered from (4). If \( \sigma \neq 0 \) the vectors \( \eta \) and \( \zeta \) are linearly independent.