Superconvergence analysis of the Galerkin FEM for a singularly perturbed convection-diffusion problem with characteristic layers

Sebastian Franz∗, Torsten Linß†

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Abstract

We analyze the superconvergence property of the Galerkin finite element method for elliptic convection-diffusion problems with characteristic layers. This method on Shishkin meshes is known to be (almost) first order accurate in the energy-norm induced by the bilinear form of the weak formulation, uniformly in the perturbation parameter. In the present paper the method is shown to be almost second order superconvergent in this energy-norm for the difference between the FEM solution and the bilinear interpolant of the exact solution. This supercloseness property is used to improve the accuracy to almost second order by means of a postprocessing procedure. Numerical experiments confirm these results.


Key words: Singular perturbation, parabolic layers, Shishkin meshes, superconvergence.

1 Introduction

We consider the singularly perturbed model convection-diffusion equation

\[
Lu := -\varepsilon \Delta u - bu_x + cu = f \quad \text{in } \Omega = (0, 1)^2, \quad (1.1a)
\]

\[
u = 0 \quad \text{on } \Gamma = \partial \Omega \quad (1.1b)
\]

Convection-diffusion problems may be regarded as linearized versions of the Navier-Stokes equations. Therefore, it is important to devise efficient numerical methods for their approximate solution. Surveys of such methods can be found in [8].

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We shall assume that the data satisfies $b \in W_1^1(\Omega)$ and $c \in L_\infty(\Omega)$. Additionally, $b \geq \beta$ on $\bar{\Omega}$ with some positive constant $\beta$, while $0 < \varepsilon \ll 1$ is a small perturbation parameter. Its presence gives rise to an exponential layer of width $O(\varepsilon)$ near the outflow boundary at $x = 0$ and to two parabolic layers of width $O(\sqrt{\varepsilon})$ near the characteristic boundaries at $y = 0$ and $y = 1$; see Fig. 1.

![Figure 1: Typical solution to (1.1) with parabolic layers (left and right) and an exponential layer (front)](image)

To ensure coercivity of the bilinear form associated with the differential operator $L$ we furthermore shall assume that

$$c + \frac{1}{2} b_x \geq \gamma > 0. \quad (1.2)$$

Therefore (1.1) possesses a unique solution in $H_0^1(\Omega)$. Note that (1.2) can always be ensured by a simple transformation $\tilde{u}(x, y) = u(x, y)e^{\kappa x}$ with $\kappa$ chosen suitably. Due to the presence of layers the use of quasi uniform meshes does not give accurate approximations of (1.1) unless the mesh size is of the order of the perturbation parameter which in practice constitutes a prohibitive restriction. Therefore layer-adapted meshes have to be used to obtain efficient discretisations. Based on a priori knowledge of the layer behaviour we shall construct piecewise uniform meshes —so called Shishkin meshes— that resolve the layers and yield robust (or uniform) convergence. On these meshes the Galerkin finite element method (GFEM) with bilinear trial and test functions will be analysed. For problems of type (1.1) with only exponential layers the GFEM on Shishkin meshes is well understood. Almost first order uniform convergence in the energy norm was established by O’Riordan and Stynes [9], while Zhang [11] and Linß [5] proved uniform superconvergence of almost second order in discrete versions of that norm.

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In the present paper problems with parabolic (or characteristic) layers will be considered. Practically these are more important. They can be considered as models of the flow past a surface. Here we shall give for the first time a rigorous superconvergence analysis of the GFEM on Shishkin meshes for convection-diffusion problems with parabolic layers. According to [7] the error of the GFEM on appropriately constructed Shishkin meshes for (1.1) satisfies

$$\| u - u^N \| \leq C N^{-1} \ln N,$$

where $N$ is the number of mesh nodes used in each coordinate direction, $C$ is a constant that is independent of the perturbation parameter $\epsilon$ and of $N$, and $\| \cdot \|_\epsilon$ is the energy norm naturally associated with the variational formulation of (1.1). We shall improve this result and show that the method is almost second order superconvergent (or superclose)

$$\| u^I - u^N \| \leq C N^{-2} \ln^2 N,$$

i.e., the difference between the numerical solution and the interpolant of the exact solution is significantly smaller. Because of this superconvergence a simple postprocessing operator $P$ can be constructed to give an enhanced approximation:

$$\| u - Pu^N \| \leq C N^{-2} \ln^2 N.$$

The results of the present paper also form the basis of the error analysis for the streamline-diffusion FEM in the forthcoming paper [3].

The paper is organized as follows. In Section 2 properly adapted Shishkin meshes are constructed based on knowledge of the layer behaviour of the solution. Section 3 gives bounds for the interpolation error which are then used in Section 4 to study the superconvergence property of the GFEM. Section 5 shows how the superconvergence property and postprocessing by biquadratic interpolation can be used to enhance the accuracy of the approximate solution. Numerical experiments that illustrate our theoretical findings are presented in Section 6.

We remark that the analysis is full of a significant number of very technical details. In most cases these are deferred to appendices. In doing so we are attempting to provide a presentation of the essential ideas and results that is easier to access.

**Notation.** Throughout $C$ denotes a generic constant that is independent of both the perturbation parameter $\epsilon$ and the number of mesh points used.

## 2 Solution decomposition and layer-adapted meshes

As mentioned before the solution $u$ of (1.1) has an exponential layer at $x = 0$ and two parabolic layers at $y = 0$ and $y = 1$. For our later analysis we shall suppose that $u$ can be split into a regular solution component and various layer parts:

**Assumption 2.1.** The solution $u$ of (1.1) can be decomposed as

$$u = v + w_1 + w_2 + w_{12},$$

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where for all $x, y \in [0, 1]$ and $0 \leq i + j \leq 2$ we have the pointwise estimates

$$
|\partial^i_x \partial^j_y v(x, y)| \leq C,
|\partial^i_x \partial^j_y w_1(x, y)| \leq C\varepsilon^{-i}e^{-\beta x/\varepsilon},
$$

(2.1a)

$$
|\partial^i_x \partial^j_y w_2(x, y)| \leq C\varepsilon^{-j/2} \left( e^{-y/\sqrt{\varepsilon}} + e^{-(1-y)/\sqrt{\varepsilon}} \right),
$$

(2.1b)

$$
|\partial^i_x \partial^j_y w_{12}(x, y)| \leq C\varepsilon^{-(i+j/2)} e^{-\beta x/\varepsilon} \left( e^{-y/\sqrt{\varepsilon}} + e^{-(1-y)/\sqrt{\varepsilon}} \right)
$$

(2.1c)

and for $0 \leq i + j \leq 3$ the $L_2$ bounds

$$
\|\partial^i_x \partial^j_y v\|_{0,\Omega} \leq C,
\|\partial^i_x \partial^j_y w_1\|_{0,\Omega} \leq C\varepsilon^{-i+1/2},
$$

(2.2a)

$$
\|\partial^i_x \partial^j_y w_2\|_{0,\Omega} \leq C\varepsilon^{-j/2+1/4},
\|\partial^i_x \partial^j_y w_{12}\|_{0,\Omega} \leq C\varepsilon^{-i-j/2+3/4}
$$

(2.2b)

and

$$
\|\partial^i_x \partial^j_y u\|_{0,\Omega} \leq C \left( 1 + \varepsilon^{-i+1/2} + \varepsilon^{-j/2+1/4} + \varepsilon^{-i-j/2+3/4} \right).
$$

(2.2c)

Remark 2.2. For $i + j \leq 2$ the $L_2$ bounds (2.2) follow from the pointwise bounds (2.1).

When discretizing (1.1), we use a piecewise uniform mesh —a so-called Shishkin mesh— with $N$ mesh intervals in both $x$- and $y$-direction which condenses in the layer regions.

For this purpose define the mesh transition parameters

$$
\lambda_x := \min \left\{ \frac{1}{2}, \frac{\sigma \varepsilon}{\beta} \ln N \right\}
\text{ and } \lambda_y := \min \left\{ \frac{1}{4}, \frac{\sigma \sqrt{\varepsilon}}{\ln N} \right\}
$$

with some positive parameter $\sigma$ that will be fixed later.

The domain $\Omega$ is divided into four (six) subregions —see Fig. 2— with $\Omega_{12}$ covering the exponential layer, $\Omega_{21}$ the parabolic layer, $\Omega_{22}$ the corner layer and $\Omega_{11}$ the remaining region which does not have layers.

<table>
<thead>
<tr>
<th>$\Omega_{22}$</th>
<th>$\Omega_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_{12}$</td>
<td>$\Omega_{11}$</td>
</tr>
</tbody>
</table>

- $\Omega_{11} := [\lambda_x, 1] \times [\lambda_y, 1 - \lambda_y]$
- $\Omega_{12} := [0, \lambda_x] \times [\lambda_y, 1 - \lambda_y]$
- $\Omega_{21} := [\lambda_x, 1] \times ([0, \lambda_y] \cup [1 - \lambda_y, 1])$
- $\Omega_{22} := [0, \lambda_x] \times ([0, \lambda_y] \cup [1 - \lambda_y, 1])$

Figure 2: Dissection of $\Omega$

These subdomains will be uniformly dissected to give the final triangulation.
For the mere sake of simplicity in our subsequent analysis we shall assume that
\[ \lambda_x = \frac{\sigma \varepsilon}{\beta} \ln N \leq \frac{1}{2} \quad \text{and} \quad \lambda_y = \sigma \sqrt{\varepsilon} \ln N \leq \frac{1}{4} \] (2.3)
as is typically the case for (1.1).

Note the mesh transition parameters \( \lambda_x \) and \( \lambda_y \) have been chosen such that the layer
terms of \( u \) are of order \( N^{-\sigma} \) on \( \Omega_{11} \), i.e.,
\[ |w_1(x, y)| + |w_2(x, y)| + |w_{12}(x, y)| \leq CN^{-\sigma} \quad \text{for} \ (x, y) \in \Omega_{11}. \]

Typically \( \sigma \) is chosen equal to the formal order of the method or to accommodate the
error analysis: We shall require \( \sigma \geq 5/2 \) for technical reasons.

In order to construct our final mesh let \( N \) be divisible by 4. Divide each of the intervals
\([0, \lambda_x]\) and \([\lambda_x, 1]\) uniformly into \( N/2 \) subintervals. We get our mesh in \( x \)-direction: \( x_i, \)
\( i = 0, \ldots, N. \) The mesh is \( y \)-direction, \( y_j, j = 0, \ldots, N, \) is obtained by uniformly dividing
\([0, \lambda_y]\) and \([1 - \lambda_y, 1]\) into \( N/4 \) subintervals and \([\lambda_y, 1 - \lambda_y]\) into \( N/2 \) subintervals. By
drawing lines through these mesh points parallel to the \( x \)-axis and \( y \)-axis the domain \( \Omega \)
is partitioned into rectangles. This triangulation is denoted by \( T^N \); see Fig. 3.

![Figure 3: Triangulation \( T^N \) of \( \Omega \)](image)

The mesh sizes \( h_i := x_i - x_{i-1} \) and \( k_j := y_j - y_{j-1} \) satisfy
\[
h_i = \begin{cases} 
    h_2 := \frac{2\sigma \varepsilon \ln N}{\beta}, & \text{for } i = 1, \ldots, N/2, \\
    h_1 := \frac{2(1 - \lambda_x)}{N}, & \text{for } i = N/2 + 1, \ldots, N 
\end{cases}
\]
and
\[
k_j = \begin{cases} 
    k_2 := 4\sigma \sqrt{\varepsilon} \ln N, & \text{for } j = 1, \ldots, N/4 \text{ and } j = 3N/4 + 1, \ldots, N \\
    k_1 := \frac{2(1 - 2\lambda_y)}{N}, & \text{for } i = N/4 + 1, \ldots, 3N/4. 
\end{cases}
\]
Note the mesh sizes $\hbar$ and $k$ satisfy
\begin{equation}
\hbar_2 \leq N^{-1} \leq \hbar_1 \leq 2N^{-1} \quad \text{and} \quad k_2 \leq N^{-1} \leq k_1 \leq 2N^{-1},
\end{equation}
properties which are essential when inverse inequalities are applied in our later analysis. For the mesh elements we shall use two notations: $\tau_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for a specific element, and $\tau$ for a generic mesh rectangle with base $h$ and height $k$.

3 The interpolation error

Problem (1.1) will be discretized by means of finite element methods. Their analysis requires precise knowledge of the interpolation error which will be provided in this section. Given any $u \in C^0(\bar{\Omega})$ and a triangulation $T^N$ of $\Omega$ into rectangles we denote by $u^I$ the nodal piecewise bilinear interpolant to $u$ over $T^N$. The main interpolation-error results can be obtained using the technique in [2, 9]. The adaptation is a straightforward, though tedious task.

**Theorem 3.1.** Let $u^I$ be the linear or bilinear interpolant of $u$ on a Shishkin mesh with $\sigma \geq 2$. Then the interpolation error satisfies
\begin{align*}
\|u - u^I\|_{L^\infty(\Omega_{11})} &\leq C N^{-2}, \\
\|u - u^I\|_{L^\infty(\Omega \setminus \Omega_{11})} &\leq C N^{-2} \ln^2 N
\end{align*}
and
\[ \|\|u - u^I\|\|_\varepsilon \leq C N^{-1} \ln N, \]
where
\[ \|v\|_\varepsilon := \varepsilon |v|_1^2 + \gamma \|v\|_0^2 \quad \text{for all } v \in H^1_0(\Omega), \]
$| \cdot |_{1,D}$ is the standard seminorm in $H^1(D)$ and $\| \cdot \|_{0,D}$ the norm in $L^2(D)$. If $D = \Omega$ then the subscript is dropped from the notation.

**Remark 3.2.** Bounds for the interpolation error in the $L_2$ norm on the various subdomains of $\Omega$ are easily obtained from the $L_\infty$ bounds:
\[ \|u - u^I\|_{0,\Omega_{k\ell}} \leq (\text{meas } \Omega_{k\ell})^{1/2} \|u - u^I\|_{\infty,\Omega_{k\ell}}, \quad k, \ell = 1, 2. \]
The proof makes use of the following anisotropic interpolation error bounds given in [1, Lemma 7.11]
\begin{align*}
\|w - w^I\|_{L_p(\tau)} &\leq C \left\{ h^2 \|w_{xx}\|_{L_p(\tau)} + k^2 \|w_{yy}\|_{L_p(\tau)} \right\} \quad \text{(3.1)}
\end{align*}
and
\begin{align*}
\|(w - w^I)_x\|_{L_p(\tau)} &\leq C \left\{ h \|w_{xx}\|_{L_p(\tau)} + k \|w_{xy}\|_{L_p(\tau)} \right\} \quad \text{(3.2)}
\end{align*}
which hold true for \( p \in [1, \infty] \) and arbitrary \( w \in W^2_p(\Omega) \). Furthermore
\[
\| (w - w)^\prime \|_{L_\infty(\tau)} \leq \| w_x \|_{L_\infty(\tau)} + \| w_x^\prime \|_{L_\infty(\tau)} \leq 2 \| w_x \|_{L_\infty(\tau)}
\] (3.3)
is used. Clearly analogous results hold true for \( (w - w)^\prime_y \).

For our later superconvergence analysis we require more detailed results for the interpolation operator applied to the various parts of the solution decomposition. These are given in Appendix A.

## 4 Galerkin FEM

The variational formulation of (1.1) is: Find \( u \in H^1_0(\Omega) \) such that
\[
a_{Gal}(u, v) := \varepsilon(\nabla u, \nabla v) - (bu_x, v) + (cu, v) = f(v) := (f, v) \quad \text{for all } v \in H^1_0(\Omega),
\] (4.1)

where \((\cdot, \cdot)_D\) denotes the standard scalar product in \( L_2(D) \). If \( D = \Omega \) we drop the \( \Omega \) from the notation.

The bilinear form \( a_{Gal}(\cdot, \cdot) \) is coercive with respect to the \( \varepsilon \)-weighted energy norm, i.e.,
\[
a_{Gal}(v, v) \geq \| v \|^2_\varepsilon \quad \text{for all } v \in H^1_0(\Omega).
\]

Because of the coercivity the Lax-Milgram Lemma ensures the existence of a unique solution \( u \in H^1_0(\Omega) \) of the variational formulation.

Let \( V^N \subset H^1_0(\Omega) \) be a finite-element space consisting of piecewise bilinear element over the Shishkin mesh. Then the discretisation is: Find \( u^N \in V^N \) such that
\[
a_{Gal}(u^N, v^N) = f(v^N) \quad \text{for all } v^N \in V^N.
\] (4.2)

The uniqueness of this solution is guaranteed by the coercivity of \( a_{Gal} \).

### Theorem 4.1 (Superconvergence)

Let \( \sigma \geq 5/2 \). Then the GFEM solution \( u^N \) satisfies
\[
\| u - u^N \|_\varepsilon \leq CN^{-2}\ln^2 N
\]

### Remark 4.2

A similar result was established in [5] for problems with exponential layers only and later slightly improved upon by the author [6]. However, due to the different nature of the parabolic layers, the term \( w_2 \) in the decomposition, the analysis from the afore mentioned papers requires a number of changes and additional new ideas.

The proof of Theorem 4.1 starts from the coercivity and Galerkin orthogonality of the bilinear form \( a_{Gal}(\cdot, \cdot) \).

\[
\| u - u^N \|_\varepsilon^2 \leq \| a_{Gal}(u - u^N, u^N) \| \\
\leq \varepsilon \left( \| (\nabla(u - u^N), \nabla(u^N)) \| \\
+ \| (b(u - u^N)_x, u^N - u^N) \| + \| (c(u - u^N), u^N - u^N) \| .
\]

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In Appendix C it will be shown that all three terms on the right-hand side can be bounded by \( CN^{-2} \ln^2 N \| u' - u^N \|_\varepsilon \). The statement of the Theorem follows upon dividing by \( \| u' - u^N \|_\varepsilon \).

The crucial ingredients in the analysis are the following integral identities from [4].

**Lemma 4.3.** Let \( \tau_{ij} \in T^N \) be an arbitrary mesh rectangle with midpoint \((\tilde{x}_i, \tilde{y}_j)\) and edges \( \ell_1, \ell_2 \) that are parallel to the y-axis.

For any function \( w \in C^3(\bar{\tau}_{ij}) \) and any bilinear function \( \chi \) there holds

\[
\int_{\tau_{ij}} (w - w') x \chi_x = \int_{\tau_{ij}} \left[ F_j \chi_x - \frac{1}{3} (F_j') \chi_{xy} \right] w_{xyy} \tag{4.3}
\]

and

\[
\int_{\tau_{ij}} (w - w') x \chi = H_{ij}(w, \chi) + \frac{h_i^2}{12} \left( \int_{\ell_1} - \int_{\ell_2} \right) \chi w_{xx} dy \tag{4.4}
\]

with

\[
H_{ij}(w, \chi) := \int_{\tau_{ij}} \left[ F_j (\chi - E_i \chi_x) - \frac{(F_j')}{3} (\chi_y - E_i \chi_{xy}) \right] w_{xyy} + \int_{\tau_{ij}} \left[ \frac{(E_i')}{6} \chi_x - \frac{h_i^2}{12} \chi \right] w_{xxx}
\]

and

\[
E_i(x) := \frac{(x - \tilde{x}_i)^2}{2} - \frac{h_i^2}{8} \quad \text{and} \quad F_j(y) := \frac{(y - \tilde{y}_j)^2}{2} - \frac{k_j^2}{8}.
\]

**Remark 4.4.** The Cauchy-Schwarz inequality and the inverse inequality (B.1) applied to (4.3) give

\[
\left\| (w - w') x \chi_x \right\|_{\tau_{ij}} \leq C k_j^2 \| w_{xyy} \|_{0, \tau_{ij}} \| \chi_x \|_{0, \tau_{ij}} \quad \text{for all} \ w \in C^3(\bar{\tau}_{ij}); \tag{4.5}
\]

similarly

\[
| H_{ij}(w, \chi) | \leq C \left\{ k_j^2 \| w_{xyy} \|_{0, \tau_{ij}} + h_i^2 \| w_{xxx} \|_{0, \tau_{ij}} \right\} \| \chi \|_{0, \tau_{ij}}. \tag{4.6}
\]

Thus both terms are formally of second order. However in the context of the present paper the uniformity with respect to \( \varepsilon \) remains to be established. Furthermore, the line integrals in (4.4) do not cancel on a strongly non-uniform mesh like the Shishkin mesh we are using and therefore require careful treatment; see Appendix C.

### 5 Postprocessing and enhancement of accuracy

Following [10], we use the superconvergence property to construct a better numerical approximation via postprocessing. Suppose \( N \) is divisible by 8 and construct a coarser
Figure 4: Macroelements $M$ of $\tilde{T}^{N/2}$ constructed from $T^N$

mesh $\tilde{T}^{N/2}$ composed of macrorectangles $M$, each consisting of four rectangles of $T^N$. Furthermore each macroelement $M$ is supposed to belong to one of the four subdomains of $\Omega$ only; see Figure 4. Let $P_M$ denote the projection/interpolation operator that maps any continuous function $v$ onto that biquadratic function $P_M v$ that coincides with $v$ at the nine mesh nodes of $T^N$ in $M$. This piecewise projection is extended to give a global continuous function by setting

$$(Pv)(x, y) := (P_M v)(x, y) \quad \text{for } (x, y) \in M.$$  

The projection $P$ enjoys the stability properties

$$P(v') = Pv \quad \text{for all } v \in C(\bar{\Omega}), \quad (5.1a)$$

and

$$|||Pv^N|||_\varepsilon \leq C |||v^N|||_\varepsilon \quad \text{for all } v^N \in V^N; \quad (5.1b)$$

see [10, Lemma 5.4].

Now let us show that $Pv^N$ better approximates $u$ than $v^N$ does. Let $\sigma \geq 5/2$. Then a triangle inequality, (5.1) and Theorem 4.1 give

$$|||Pv^N - u|||_\varepsilon \leq |||P(u^N - u^I)|||_\varepsilon + |||Pv - u|||_\varepsilon \leq CN^{-2}\ln^2 N + |||Pv - u|||_\varepsilon \leq CN^{-2}\ln^2 N + |||Pv - u|||_\varepsilon$$

For the error of the biquadratic interpolation we have the bound

$$|||Pv - u|||_\varepsilon \leq C (\varepsilon N^{-\sigma+1} + N^{-2}\ln^2 N); \quad (5.2)$$

see Appendix D. We get

**Theorem 5.1.** Let $u^N$ be the numerical solution obtained by the GFEM on a Shishkin mesh with $\sigma \geq 5/2$. Then the postprocessed solution satisfies

$$|||Pv^N - u|||_\varepsilon \leq C (\varepsilon N^{-\sigma+1} + N^{-2}\ln^2 N).$$
Table 1: Problem I

Remark 5.2. Theorem 5.1 shows that a simple postprocessing procedure can improve the uniform convergence to almost second order if \( \varepsilon \leq C N^{-1/2} \ln^2 N \), which in practice does not constitute a major restriction, or if in the construction of the Shishkin mesh \( \sigma \geq 3 \) is chosen.

The same postprocessing can be applied to enhance the accuracy of the SDFEM analysed in [3] since that method also enjoys the superconvergence property of Theorem 4.1.

6 Numerical Results

Two different test problems will be studied. In our experiments the uniform errors are estimated by taking the maximum over various values of \( \varepsilon \), i.e.

\[
\varepsilon^*_N = \max_{\varepsilon = 10^{-2}, 10^{-4}, \ldots, 10^{-10}} \| u - u^N \|_\infty.
\]

The rate of convergence is estimated using the formula

\[
r^*_N = \log \left( \frac{\varepsilon^*_N}{\varepsilon^{2N}_*} \right).
\]

Problem I is given by

\[
-\varepsilon \Delta u - (2 - x)u_x + \frac{3}{2} u = f \quad \text{in} \quad \Omega = (0, 1)^2, \quad u|_{\partial \Omega} = 0
\]

with homogeneous Dirichlet boundary conditions and right-hand side \( f \) chosen such that

\[
u(x, y) = \left( \cos \frac{\pi x}{2} - \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} \right) \left( \frac{1 - e^{-y/\sqrt{\varepsilon}}}{1 - e^{-1/\sqrt{\varepsilon}}} \right) \left( 1 - e^{-(1-y)/\sqrt{\varepsilon}} \right) \]

is the exact solution. Table 1 displays the errors in various norms of the GFEM for test problem I. They are clear illustrations of the first order convergence (1.3), the almost second order superconvergence result of Theorem 4.1 and the second order convergence result of the postprocessed solution of Theorem 5.1. The last column of the table gives the error estimates in the \( L_\infty \) norm. We observe almost second order convergence, although we do not have theoretical justification for this behaviour.
Table 2: Problem II

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|u - u^N|_\infty$ rate</th>
<th>$|u - Pu^N|_\infty$ rate</th>
<th>$|u^l - u^N|_\varepsilon$ rate</th>
<th>$|u - u^N|_{L\infty}$ rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.127e-02 0.70</td>
<td>6.601e-03 1.70</td>
<td>3.665e-03 1.42</td>
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<td>1.698e-03 1.65</td>
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<tr>
<td>1024</td>
<td>1.382e-03</td>
<td>1.528e-05</td>
<td>1.508e-05</td>
<td>5.422e-04</td>
</tr>
</tbody>
</table>

Problem II.

$$-\varepsilon \Delta u - u_x + u = f \quad \text{in } \Omega = (0, 1)^2, \quad u|_{\partial \Omega} = 0$$

with right-hand side $f$ chosen such that

$$u = \sin(\pi x) \frac{(1 - e^{-y/\sqrt{\varepsilon}}) (1 - e^{-(1-y)/\sqrt{\varepsilon}})}{1 - e^{-1/\sqrt{\varepsilon}}}$$

is the solution. This problem has been selected because it exhibits only parabolic layers the layer we are particularly concerned with. The construction of the Shishkin mesh is changed by using a uniform mesh in $x$-direction since there is no layer at $x = 0$.

Table 2 gives the errors of the method. The behaviour is similar to Problem I: almost first order convergence and second order superconvergence in the energy norms.

References


A Detailed interpolation error bounds

In this first appendix we derive interpolation error bounds for the various parts of the decomposition $u = v + w_1 + w_2 + w_{12}$ that will be used later when giving details of the superconvergence analysis. First, (3.2) and (2.2) give

$$ \| (v - v^I)_x \|_0 + \| (v - v^I)_y \|_0 \leq C N^{-1}. \tag{A.1} $$

Next, use (3.3) and (2.1) to obtain

$$ \| (w_2 - w_2^I)_x \|_{L^\infty(\Omega_{11} \cup \Omega_{12})} + \varepsilon \| (w_{12} - w_{12}^I)_x \|_{L^\infty(\Omega_{12})} \leq C N^{-\sigma} \tag{A.2a} $$

and

$$ \varepsilon \| (w_{12} - w_{12}^I)_x \|_{L^\infty(\Omega_{11})} \leq C N^{-2\sigma}. \tag{A.2b} $$

Finally bounds for the interpolation error of the layer terms are needed.

**Proposition A.1.** Let $w = w_1 + w_2 + w_{12}$. Then

$$ \| w - w^I \|_{0, \Omega_{11}} \leq C \left( \varepsilon^{1/4} N^{-\sigma} + N^{-\sigma-1/2} \right) \tag{A.3a} $$

$$ \| w - w^I \|_{0, \Omega_{12}} \leq C \varepsilon^{1/2} \left( N^{-2} \ln^2 N + N^{-\sigma} \ln^{1/2} N \right) \tag{A.3b} $$

$$ \| w - w^I \|_{0, \Omega_{21}} \leq C \varepsilon^{1/4} \left( N^{-2} \ln^2 N + N^{-\sigma} \ln^{1/2} N \right) \tag{A.3c} $$

$$ \| w - w^I \|_{0, \Omega_{22}} \leq C \varepsilon^{1/2} N^{-2} \ln^2 N \tag{A.3d} $$
**Proof.** (i) Let us first consider \( \| w - w^I \|_{0, \Omega_{11}} \). Clearly
\[
\| w - w^I \|_{0, \Omega_{11}} \leq \| w \|_{0, \Omega_{11}} + \| w^I \|_{0, \Omega_{11}}.
\]
A direct calculation and (2.1) give
\[
\| w \|_{0, \Omega_{11}} \leq C \varepsilon^{1/4} N^{-\sigma}.
\]
When bounding \( w^I \) we follow an idea by Zhang [11]. This enables us to assume \( \sigma \geq 5/2 \) rather than \( \sigma \geq 3 \) in our later analysis.

The domain \( \Omega_{11} \) is divided into \( S = [\lambda_x + h_1, 1] \times [\lambda_y + k_1, 1 - \lambda_y - k_1] \) and \( \Omega_{11} \setminus S \).

Note that \( \Omega_{11} \setminus S \) consists of only a single ply of \( O(N) \) mesh elements adjacent to the boundary layer regions. Thus
\[
\| w^I \|_{0, \Omega_{11} \setminus S} \leq \sum_{\tau \in \Omega_{11} \setminus S} h_1 k_1 \| w^I \|_{L_\infty(\tau)}^2 \leq CN^{-1} \| w \|_{L_\infty(\Omega_{11})}^2 \leq CN^{-2\sigma - 1},
\]
by (2.4) and (2.1).

For \( \tau_{ij} \in S \) we have
\[
\| w^I \|_{0, \tau_{ij}}^2 \leq h_1 k_1 \| w^I \|_{L_\infty(\tau_{ij})}^2
\leq C h_1 k_1 \left( e^{-2\beta x_{i-1}/\varepsilon} + \left( e^{-2y_{j-1}/\sqrt{\varepsilon}} + e^{-2(1-y_j)/\sqrt{\varepsilon}} \right) \left( 1 + e^{-2\beta x_{i-1}/\varepsilon} \right) \right)
\leq C \left\{ \int_{\tau_{i-1,j-1}} e^{-2\beta x_i/\varepsilon} + e^{-2y_{j-1}/\sqrt{\varepsilon}} \left( 1 + e^{-2\beta x_i/\varepsilon} \right) \right. + \left. \int_{\tau_{i-1,j}} e^{-2(1-y_j)/\sqrt{\varepsilon}} \left( 1 + e^{-2\beta x_i/\varepsilon} \right) \right\}
\]
Summing over \( S \), we obtain
\[
\| w^I \|_{0,S}^2 \leq C \int_{\Omega_{11}} e^{-2\beta x_i/\varepsilon} + \left( e^{-2y_{j-1}/\sqrt{\varepsilon}} + e^{-2(1-y_j)/\sqrt{\varepsilon}} \right) \left( 1 + e^{-2\beta x_i/\varepsilon} \right) \leq C \varepsilon^{1/2} N^{-2\sigma},
\]
which together with (A.4) completes the proof of (A.3a).

(ii) On \( \Omega_{12} \) use (3.1) and (2.2) to establish
\[
\| w_1 - w_1^I \|_{0, \Omega_{12}} \leq C \varepsilon^{1/2} N^{-2} \ln^2 N,
\]
while for \( w_2 \) and \( w_{12} \) we proceed as follows
\[
\| w_2 + w_{12} - (w_2 + w_{12})^I \|_{0, \Omega_{12}} \leq C (\text{meas } \Omega_{12})^{1/2} \| w_2 + w_{12} \|_{L_\infty(\Omega_{12})} \leq C \varepsilon^{1/2} N^{-\sigma} \ln^{1/2} N.
\]
We get (A.3b).
(iii) On $\Omega_{21}$ we employ (3.1) to bound the error in $w_2$:
\[
\|w_2 - w_2^f\|_{0, \Omega_{21}} \leq C\varepsilon^{1/4} N^{-2} \ln^2 N.
\]
On the other hand
\[
\|w_1 + w_{12} - (w_1 + w_{12})^f\|_{0, \Omega_{21}}
\leq (\text{meas } \Omega_{21})^{1/2} \|w_1 + w_{12}\|_{L_\infty(\Omega_{21})} \leq C\varepsilon^{1/4} N^{-\sigma} \ln^{1/2} N.
\]
This is (A.3c).

(iv) Finally, we study $w - w^f$ on $\Omega_{22}$. Ineq. (3.1) gives
\[
\|w - w^f\|_{0, \Omega_{22}} \leq CN^{-2} \ln^2 N \left\{ \varepsilon^2 \|w\|_{0, \Omega_{22}} + \varepsilon^{3/2} \|w_{xy}\|_{0, \Omega_{22}} + \varepsilon \|w_{yy}\|_{0, \Omega_{22}} \right\}
\leq C\varepsilon^{3/4} N^{-2} \ln^{5/2},
\]
by (2.1) and a direct calculation. Using (2.3), we obtain (A.3d).

B Inverse estimates

Throughout the remaining analysis we shall make frequent use of the following inverse estimates. Let $\chi$ be a polynomial on the mesh rectangle $\tau$. Then
\[
\|\chi_x\|_{L_p(\tau)} \leq Ch^{-1}\|\chi\|_{L_p(\tau)} \quad \text{and} \quad \|\chi_y\|_{L_p(\tau)} \leq Ck^{-1}\|\chi\|_{L_p(\tau)} \quad \text{(B.1)}
\]
\[
\int_{y_{j-1}}^{y_j} \left|\chi(x, y)\right| \, dy \leq Ch^{-1}_i\|\chi\|_{L_1(\tau_i)} \quad \text{(B.2)}
\]
\[
\|\chi\|_{L_q(\tau)} \leq C(\text{meas } \tau)^{1/q-1/p}\|\chi\|_{L_p(\tau)} \quad \text{for } p, q \in [1, \infty] \quad \text{(B.3)}
\]
where $h$ is the base and $k$ the height of $\tau$.

C Proof of Theorem 4.1

Proposition C.1. Let $\sigma \geq 2$. Then the solution $u^N$ of the Galerkin discretization (4.2) satisfies
\[
\varepsilon \left| \langle \nabla (u - u^f), \nabla \chi \rangle \right| \leq CN^{-2} \ln^2 N \|\chi\|_e \quad \text{for all } \chi \in V^N.
\]

Proof. Summing (4.5) over all elements in $\Omega_{k\ell}$, one of our four subdomains of $\Omega$, and using a discrete Cauchy-Schwarz inequality, we get
\[
\left| \langle (w - w^f)x, \chi_x \rangle_{\Omega_{k\ell}} \right| \leq Ck^2\|w_{xy}\|_{0, \Omega_{k\ell}}\|\chi_x\|_{0, \Omega_{k\ell}} \quad \text{for } k, \ell = 1, 2 \quad \text{(C.1)}
\]
and—because of symmetry—

\[
\left| (w - w^I)_y, \chi_y \right|_{\Omega_{k\ell}} \leq C\hbar^2 \|w_x\|_{0,\Omega_{k\ell}} \|\chi_y\|_{0,\Omega_{k\ell}} \quad \text{for } k, \ell = 1, 2. \tag{C.2}
\]

(i) First let us consider \((u - u^I)_x, \chi_x\). Inequalities (C.1) and (2.2) yield

\[
\varepsilon \left| (u - u^I)_x, \chi_x \right|_{\Omega_{22}\cup\Omega_{12}} \leq C \varepsilon^{1/4} N^{-2} \ln^2 N \|\chi\|_\varepsilon \tag{C.3}
\]

and—recalling the decomposition \(u = v + w_1 + w_2 + w_{12}\)—

\[
\varepsilon \left| (v + w_1) - (v + w_1^I)_x, \chi_x \right|_{\Omega_{12}\cup\Omega_{11}} \leq CN^{-2} \|\chi\|_\varepsilon. \tag{C.4}
\]

because \(k_2 = 4 \varepsilon^{1/2} \sigma N^{-1} \ln N\) and \(k_1 \leq 2N^{-1}\).

The Hölder and the Cauchy-Schwarz inequalities yield

\[
\left| (w - w^I)_x, \chi_x \right|_D \leq \|(w - w^I)_x\|_{L_\infty(D)} \|\chi_x\|_{L_1(D)} \\
\leq \|(w - w^I)_x\|_{L_\infty(D)} \left(\text{meas}(D)\right)^{1/2} \|\chi_x\|_{0,D}.
\]

From this inequality and (A.2), we obtain

\[
\varepsilon \left| ((w_2 + w_{12}) - (w_2 + w_{12}^I)_x, \chi_x)_{\Omega_{12}} \right| \leq CN^{-\sigma} \ln^{1/2} N \|\chi\|_\varepsilon \tag{C.5}
\]

and

\[
\varepsilon \left| ((w_2 - w^I_2)_x, \chi_x)_{\Omega_{11}} \right| \leq C \varepsilon^{1/2} N^{-\sigma} \|\chi\|_\varepsilon, \tag{C.6}
\]

because \(\text{meas}(\Omega_{12}) \leq \sigma \varepsilon\beta^{-1} \ln N\) and \(\text{meas}(\Omega_{11}) \leq 1\).

Next consider the corner layer term \(w_{12}\) on \(\Omega_{11}\). Starting again from the Hölder inequality, we get

\[
\varepsilon \left| (w_{12} - w_{12}^I)_x, \chi_x \right|_{\Omega_{11}} \leq \varepsilon \|(w_{12} - w_{12}^I)_x\|_{L_\infty(\Omega_{11})} \|\chi_x\|_{0,\Omega_{11}} \leq CN^{-2\alpha+1} \|\chi\|_\varepsilon, \tag{C.7}
\]

by (A.2), the inverse inequality (B.1) and \(h_1 \geq N^{-1}\).

Collecting (C.3)–(C.7), we get

\[
\varepsilon \left| (u - u^I)_x, \chi_x \right|_{\Omega} \leq CN^{-2} \ln^2 N \|\chi\|_\varepsilon. \tag{C.8}
\]

(ii) In the second part of the proof we study \((u - u^I)_y, \chi_y\). Similar to the first part of the analysis, (C.2) and (2.2) give

\[
\varepsilon \left| (u - u^I)_y, \chi_y \right|_{\Omega_{22}\cup\Omega_{12}} \leq C \varepsilon^{3/4} N^{-2} \ln^2 N \|\chi\|_\varepsilon \tag{C.9}
\]
and

\[ \varepsilon \left| \left((v + w_2) - (v + w_2)^{T}\right)_y, \chi_y\right|_{\Omega_1, \Omega_1} \leq C \varepsilon^{1/4} N^{-2} \|\chi\|_\varepsilon, \]  

(C.10)

because \( h_2 = 2\varepsilon \sigma \beta^{-1} N^{-1} \ln N \) and \( h_1 \leq 2N^{-1} \).

Imitate the argument that lead to (C.5) and (C.6) to obtain

\[ \varepsilon \left| \left((w_1 + w_{12}) - (w_1 + w_{12})^{T}\right)_y, \chi_y\right|_{\Omega_2} \leq C \varepsilon^{1/4} N^{-\sigma} \ln^{1/2} N \|\chi\|_\varepsilon \]  

(C.11)

and

\[ \varepsilon \left| \left((w_1 - w_1^{T}\right)_y, \chi_y\right|_{\Omega_1} \leq C \varepsilon^{1/2} N^{-\sigma} \|\chi\|_\varepsilon, \]  

(C.12)

since \( \text{meas}(\Omega_2) \leq 2\varepsilon \varepsilon^{1/2} \ln N \) and \( \text{meas}(\Omega_1) \leq 1 \).

Finally, adapting the argument for (C.7), we get

\[ \varepsilon \left| \left((w_{12} - w_{12}^{T}\right)_y, \chi_y\right|_{\Omega_1} \leq \varepsilon \|\left(w_{12} - w_{12}^{T}\right)_y\|_{L_\infty(\Omega_1)} \|\chi_y\|_{0, \Omega_1} \leq C \varepsilon^{1/2} N^{-2\sigma + 1} \|\chi\|_\varepsilon, \]  

(C.13)

Collect (C.9)–(C.13). We get

\[ \varepsilon \|\left(u - u^{T}\right)_y, \chi_y\|_{\Omega} \leq C \varepsilon^{1/4} N^{-2} \|\chi\|_\varepsilon. \]  

Combine this with (C.8) to complete the proof. \( \square \)

**Proposition C.2.** Let \( \sigma \geq 2 \). Then

\[ \left| (c(u - u^{T}), \chi) \right| \leq CN^{-2} \ln^2 N \|\chi\|_\varepsilon \quad \text{for all } \chi \in \mathcal{V}^N. \]

**Proof.** This follows readily from the Cauchy-Schwarz inequality and Theorem 3.1. \( \square \)

The most complicated term to bound is \( (b(u - u^{T}), \chi) \). Recalling the decomposition

\[ u = v + w_1 + w_2 + w_{12}, \]  

we shall study the various components of \( u \) separately. Let \( \tilde{w} = w_1 + w_{12} \). Integration by parts yields

\[ (b(u - u^{T}), \chi) = (b(v - v^{T}), \chi) + (b(w_2 - w_2^{T}), \chi) \big|_{\Omega_2} \big|_{\Omega_2} \]  

- \((b_\varepsilon(\tilde{w} - \tilde{w}^{T}), \chi) - (b_\varepsilon(\tilde{w} - \tilde{w}^{T}), \chi) \big|_{\Omega_2} \big|_{\Omega_2} \]  

- \((b_\varepsilon(w_2 - w_2^{T}), \chi) \big|_{\Omega_2} \big|_{\Omega_2} - (b_\varepsilon(w_2 - w_2^{T}), \chi) \big|_{\Omega_2} \big|_{\Omega_2} \]  

since \( \chi \in \mathcal{V}^N \) vanishes on \( \Gamma \). For the terms on the right-hand side we have the following estimates which will be proved next.

\[ \left| (b_\varepsilon(\tilde{w} - \tilde{w}^{T}), \chi) \right| \leq C \varepsilon N^{-2} \ln^2 N \|\chi\|_\varepsilon \]  

(C.14)

\[ \left| (b_\varepsilon(w_2 - w_2^{T}), \chi) \right| \leq C \varepsilon N^{-2} \ln^2 N \|\chi\|_\varepsilon, \]  

(C.15)

\[ \left| (b(v - v^{T}), \chi) \right| \leq C \varepsilon N^{-2} \ln^2 N \|\chi\|_\varepsilon, \]  

(C.16)

\[ \left| (b(w_2 - w_2^{T}), \chi) \right| \leq C \varepsilon^{1/4} N^{-2} \ln^2 N \|\chi\|_\varepsilon, \]  

(C.17)

Combining these bounds, we obtain
Proposition C.3. Let \( \sigma \geq 5/2 \). Then
\[
\left| \left( b(u - u^I)x, \chi \right) \right| \leq CN^{-2} \ln^2 N \left\| \chi \right\|_e \text{ for all } \chi \in V^N.
\]
Propositions C.1–C.3 provide the bounds required in the proof of Theorem 4.1.

**Proof of (C.14).** The interpolation error bounds of Theorem 3.1 give
\[
\left| (b_x(w_2 - w_2^I), \chi) \right| \leq C \left( \left\| w_2 - w_2^I \right\|_{0, \Omega_{12} \cup \Omega_{11}} + \left\| \tilde{w} - \tilde{w}^I \right\|_{0, \Omega_{11}} \right) \chi \leq CN^{-2} \ln^2 N \left\| \chi \right\|_e.
\]
**Proof of (C.15).**
\[
\left| (b(w_2 - w_2^I), \chi) \right| \leq C \left( \left\| w_2 - w_2^I \right\|_{0, \Omega_{11}} \chi + \left\| w_2 - w_2^I \right\|_{0, \Omega_{12}} \chi \right) \leq C N^{-2} \epsilon^{1/4} N^{-2} \ln^2 N \chi \leq CN^{-2} \ln^2 N \left\| \chi \right\|_e.
\]
by Proposition A.1.

**Proof of (C.16) and of (C.17).** Let \( b_{ij} := b(x_i, y_j) \) and define a piecewise constant approximation of \( b \) by \( \tilde{b} \equiv b_{ij} \) on \( \tau_{ij} \). Then
\[
(b(v - v^I)x, \chi) = \sum_{\tau_{ij}} \left[ b_{ij} \left( v - v^I \right)_x, \chi \right]_{\tau_{ij}} + \left( b - b_{ij} \right) (v - v^I)x, \chi \right]_{\tau_{ij}}
\]
\[
= \sum_{\tau_{ij}} b_{ij} H_{ij}(v, \chi) + \left( b - \tilde{b} \right) (v - v^I)x, \chi \right]_{\tau_{ij}} + \frac{1}{12} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( b_{i+1,j} h_{i+1}^2 - b_{ij} h_i^2 \right) \int_{y_j}^{y_{j+1}} (\chi v_{xx})(x_i, y) dy
\]
\[
=: I_1 + I_2 + I_3,
\]
by Lemma 4.3.

For \( I_1 \) inequality (4.6), (2.4) and (2.2) yield
\[
|I_1| \leq CN^{-2} \left[ \left\| v_{xxx} \right\|_0 + \left\| v_{xyy} \right\|_0 \right] \chi \leq CN^{-2} \left\| \chi \right\|_0.
\]
while Taylor expansions give

$$ ||\vec{b} - \bar{b}||_{L_{\infty}(\tau)} \leq C(h_i + k_i) \leq CN^{-1}. $$

Thus

$$ |I_2| \leq CN^{-1}||v - v^I||_0||\chi||_0 \leq CN^{-2} |||\chi||_\epsilon, $$

by the Cauchy-Schwarz inequality and (A.1).

When studying $I_3$ for $i \leq N/2$, we use

$$ \int_{y_{j-1}}^{y_j} (\chi v_{xx})(x_i, y)dy = \sum_{k=1}^{\tau_{ij}} (\chi v_{xx} + \chi v_{xxx}) $$

which implies

$$ \sum_{i=1}^{N/2} (b_{i+1,j}h_{i+1}^2 - b_{ij}h_i^2) \int_{y_{j-1}}^{y_j} (\chi v_{xx})(x_i, y)dy $$

$$ = \sum_{i=1}^{N/2} (b_{N/2+1,j}h_{N/2+1}^2 - b_{ij}h_i^2) \int_{\tau_{ij}} (\chi v_{xx} + \chi v_{xxx}). $$

For $i > N/2$ we employ

$$ \left| \int_{y_{j-1}}^{y_j} (\chi v_{xx})(x_i, y)dy \right| \leq CN\|v_{xx}\|_{L_{\infty}(\tau)}||\chi||_{L_1(\tau)}; $$

by (B.2) and (2.4).

Furthermore $|b_{i+1,j} - b_{ij}| \leq Ch_{i+1}$ and $h_i = h_{i+1} \leq 2N^{-1}$ for $i > N/2$. Thus

$$ |I_3| \leq CN^{-2} \left( \|v_{xx}\chi_x + v_{xxx}\chi\|_{L_1(\Omega_{22} \cup \Omega_{12})} + \|v_{xx}\|_{L_{\infty}} \|\chi\|_{L_1(\Omega_{21} \cup \Omega_{11})} \right) $$

$$ \leq CN^{-2} \left( \|v_{xx}\|_{L_{\infty}} (\text{meas } \Omega_{22} \cup \Omega_{12})^{1/2} \|\chi\|_{0,\Omega_{22} \cup \Omega_{12}} + \|v_{xxx}\|_{0,\Omega_{22} \cup \Omega_{12}} \right) $$

$$ \leq CN^{-2} \ln^{1/2} N \|\chi\|_\epsilon, \text{ by (2.2).} $$

Combining the estimates for the $I$'s, we get (C.16).

To prove (C.17) proceed in a similar manner:

$$ (b(w_2 - w^I_2)_x, \chi)_{\Omega_{21} \cup \Omega_{22}} $$

$$ = \sum_{\tau_{ij} \subset \Omega_{21} \cup \Omega_{22}} b_{ij}H_{ij}(w_2, \chi) + ((b - \bar{b})(w_2 - w^I_2)_x, \chi)_{\Omega_{21} \cup \Omega_{22}} $$

$$ + \frac{1}{12} \sum_{i=1}^{N-1} \left( \sum_{j=1}^{N/4} + \sum_{j=3N/4+1}^{N} \right) (b_{i+1,j}h_{i+1}^2 - b_{ij}h_i^2) \int_{y_{j-1}}^{y_j} (\chi w_{2,xx})(x_i, y)dy $$

$$ =: J_1 + J_2 + J_3 $$

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Using (4.6) and (2.2), we get
\[
|J_1| \leq C \left[ N^{-2} \|w_{2,xxx}\|_0 + k_1^2 \|w_{2,xyy}\|_0 \right] \|\chi\|_{0,\tau_{ij}} \leq C \varepsilon^{1/4} N^{-2} \ln^2 N \|\chi\|_0.
\]

For \(J_2\) an argument similar to that for \(I_2\) gives
\[
|J_2| = \left| \left( (b - \bar{b})(w_2 - w_2',) \right)_x \chi \right|_{\Omega_{21} \cup \Omega_{22}} \leq C N^{-1} \|w_2 - w_2'\|_0, \Omega_{22} \cup \Omega_{21} \|\chi\|_0
\]
\[
\leq C N^{-2} \left( \|w_{2,xx}\|_{0,\Omega_{22} \cup \Omega_{21}} + \varepsilon^{1/2} \ln N \|w_{2,xy}\|_{0,\Omega_{22} \cup \Omega_{21}} \right) \|\chi\|_0,
\]
by (3.2). Recalling (2.2), we get
\[
|J_2| \leq C \varepsilon^{1/4} N^{-2} \ln N \|\|\chi\||\varepsilon.
\]

Finally, for \(J_3\) we have, similar to (C.19)
\[
|J_3| \leq C N^{-2} \left( \|w_{2,xx}\|_{L_\infty} (\text{meas } \Omega_{22})^{1/2} \|\chi_x\|_{0,\Omega_{22}} + \|w_{2,xxx}\|_0 \|\chi\|_{0,\Omega_{22}}
\]
\[
+ \|w_{2,xx}\|_{L_\infty} (\text{meas } \Omega_{21})^{1/2} \|\chi\|_{0,\Omega_{21}} \right)
\]
\[
\leq C \varepsilon^{1/4} N^{-2} \ln N \|\|\chi\||\varepsilon, \text{ by (2.2)}.
\]

The bounds for \(J_1, J_2\) and \(J_3\) give (C.17).

### D Biquadratic interpolation error bounds

This final appendix contains the proof of (5.2). Let \(\sigma \geq 5/2\).

We use the anisotropic interpolation error bounds for biquadratic interpolation from [1, Lemma 7.11]:
\[
\|w - P w\|_{L_p(\tau)} \leq C \left\{ h_3^3 \|w_{xxx}\|_{L_p(\tau)} + k^3 \|w_{yy}\|_{L_p(\tau)} \right\} \quad (D.1a)
\]
and
\[
\|w - P w\|_{x,L_p(\tau)} \leq C \left\{ h_2^2 \|w_{xxx}\|_{L_p(\tau)} + k^2 \|w_{xy}\|_{L_p(\tau)} \right\}, \quad (D.1b)
\]
which hold true for \(p \in [1, \infty]\) and arbitrary \(w \in W^3_p(\Omega)\). Furthermore
\[
\|(Pw)_x\|_{L_\infty(\tau)} \leq C \|w_x\|_{L_\infty(\tau)}, \quad (D.2)
\]
which can be verified by a direct calculation. Clearly analogous estimates hold true for the derivatives with respect to \(y\).

Recalling the decomposition \(u = v + w_1 + w_2 + w_{12}\), we split the proof of (5.2) into four steps.

(i) For the regular solution component \(v\) use (D.1), (2.2) and the fact that the maximum mesh size is smaller than \(2N^{-1}\) to show
\[
\|P v - v\| \leq C \left( \varepsilon^{1/2} N^{-2} + N^{-3} \right).
\]
For $w_1$ on $\Omega_{12} \cup \Omega_{22}$ the same chain of arguments yields

$$\|Pw_1 - w_1\|_{\varepsilon, \Omega_{12} \cup \Omega_{22}} \leq CN^{-2} \ln^2 N,$$

since $h \leq C\varepsilon N^{-1} \ln N$ and $k \leq 2N^{-1}$ for all $\tau \subset \Omega_{12} \cup \Omega_{22}$.

On $\Omega_{21} \cup \Omega_{11}$ use a triangle inequality

$$\varepsilon^{1/2} \| (Pw_1 - w_1)_x \|_{0, \Omega_{21} \cup \Omega_{11}} \leq \varepsilon^{1/2} \| (Pw_1)_x \|_{0, \Omega_{21} \cup \Omega_{11}} + \varepsilon^{1/2} \| w_1, x \|_{0, \Omega_{21} \cup \Omega_{11}},$$

and bound the two terms separately. The second term is easily bounded using (2.1)

$$\varepsilon^{1/2} \| w_1, x \|_{0, \Omega_{21} \cup \Omega_{11}} \leq CN^{-\sigma},$$

whereas the first term needs a more detailed analysis. For any $M \subset \Omega_{21} \cup \Omega_{11}$ the inverse inequality (B.1) and the $L_\infty$-stability of $P$ give

$$\| (Pw_1)_x \|_{0, M} \leq CN \| Pw_1 \|_{0, M} \leq CN (\text{meas } M)^{1/2} \| Pw_1 \|_{L_\infty(M)}.$$

Thus

$$\| (Pw_1)_x \|_{0, \Omega_{21} \cup \Omega_{11}}^2 \leq CN \sum_{i=N/4}^{N/2-1} e^{-2\beta x_{2i}/\varepsilon} \leq CN \left( N^{-2\sigma} + \sum_{i=N/4+1}^{N/2-1} e^{-2\beta x_{2i}/\varepsilon} \right),$$

by (2.1) and the choice of the transition point. For $i > N/4$ we have

$$e^{-2\beta x_{2i}/\varepsilon} = \frac{1}{x_{2i} - x_{2(i-1)}} \int_{x_{2(i-1)}}^{x_{2i}} e^{-2\beta x_{2i}/\varepsilon} dx \leq CN \int_{x_{2(i-1)}}^{x_{2i}} e^{-2\beta x/\varepsilon} dx.$$

Hence

$$\sum_{i=N/4+1}^{N/2-1} e^{-2\beta x_{2i}/\varepsilon} \leq C\varepsilon N^{-2\sigma+1}.$$

Combining these estimates, we get

$$\| (Pw_1)_x \|_{0, \Omega_{21} \cup \Omega_{11}} \leq C(N^{-\sigma+1/2} + \varepsilon^{1/2} N^{-\sigma+1})$$

and then

$$\varepsilon^{1/2} \| (Pw_1 - w_1)_x \|_{0, \Omega_{21} \cup \Omega_{11}} \leq CN^{-\sigma} + \varepsilon N .$$

The analysis of $(Pw_1 - w_1)_y$ is simpler because the $y$-derivative of $w_1$ is bounded independent of $\varepsilon$.

$$\| (Pw_1 - w_1)_y \|_{0, \Omega_{21} \cup \Omega_{11}} \leq \| (Pw_1 - w_1)_y \|_{L_\infty(\Omega_{21} \cup \Omega_{11})} \leq C \| w_1, y \|_{L_\infty(\Omega_{21} \cup \Omega_{11})} \leq CN^{-\sigma},$$

by (D.2) and (2.1).
Similarly, we get
\[ \| Pw_1 - w_1 \|_{0, \Omega_{21} \cup \Omega_{11}} \leq \| w_1 \|_{L_\infty(\Omega_{21} \cup \Omega_{11})} \leq CN^{-\sigma}. \]

Collect all these bounds to obtain
\[ \| Pw_1 - w_1 \|_\varepsilon \leq C \left( N^{-2} \ln^2 N + \varepsilon N^{-\sigma + 1} \right). \]

(iii) When bounding \( Pw_2 - w_2 \), one can use again the above technique to give
\[ \| Pw_2 - w_2 \|_{\varepsilon, \Omega_{21} \cup \Omega_{22}} \leq C\varepsilon^{1/4} N^{-2} \ln^2 N, \]
\[ \varepsilon^{1/2}\| (Pw_2 - w_2)_x \|_{0, \Omega_{12} \cup \Omega_{11}} \leq C\varepsilon^{1/2} N^{-\sigma}, \]
\[ \varepsilon^{1/2}\| (Pw_2 - w_2)_y \|_{0, \Omega_{12} \cup \Omega_{11}} \leq CN^{-\sigma} \]
and
\[ \| Pw_2 - w_2 \|_{0, \Omega_{12} \cup \Omega_{11}} \leq CN^{-\sigma}. \]

Therefore
\[ \| Pw_2 - w_2 \|_\varepsilon \leq C(\varepsilon^{1/4} N^{-2} \ln^2 N + N^{-\sigma}) \]

(iv) Let \( \Omega^* := \Omega \setminus \Omega_{22} \). Then using known arguments, we obtain the bounds
\[ \| Pw_{12} - w_{12} \|_{\varepsilon, \Omega_{22}} \leq C\varepsilon^{1/4} N^{-2} \ln^2 N, \]
\[ \varepsilon^{1/2}\| (Pw_{12} - w_{12})_x \|_{\Omega^*} \leq C(N^{-2} + \varepsilon N^{-\sigma + 1}), \]
\[ \varepsilon^{1/2}\| (Pw_{12} - w_{12})_y \|_{\Omega^*} \leq CN^{-\sigma} \]
and
\[ \| Pw_{12} - w_{12} \|_{0, \Omega^*} \leq CN^{-\sigma}. \]

These give
\[ \| Pw_{12} - w_{12} \|_\varepsilon \leq C \left( N^{-2} \ln^2 N + \varepsilon N^{-\sigma + 1} \right). \]

Finally, combine the bounds for the four parts of the decomposition to get (5.2).