Superconvergence analysis of Galerkin FEM and SDFEM for elliptic problems with characteristic layers

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Abstract

In this paper we analyze the superconvergence property of Galerkin and streamline diffusion finite element method (SDFEM) in the case of elliptic problems with characteristic layers. For the SDFEM we give optimal parameter for maximal stability in the induced streamline diffusion norm. In the parabolic boundary layer we are able to show that the SDFEM-parameter $\delta$ can be chosen of order $\delta = C\varepsilon^{-1/4}N^{-2}$ which is confirmed by numerical results.


1 Introduction

Consider the model convection-diffusion equation

$$Lu := -\varepsilon \Delta u - bu_x + cu = f \quad \text{in } \Omega = (0, 1)^2,$$

subject to Dirichlet boundary conditions

$$u = 0 \quad \text{on } \Gamma = \partial \Omega$$

with $b \in W^1_\infty(\Omega)$, $c \in L_\infty(\Omega)$, $b \geq \beta$ on $\bar{\Omega}$ with a positive constant $\beta$, while $0 < \varepsilon \ll 1$ is a small perturbation parameter. Its presence gives rise to an exponential layer of width $O(\varepsilon)$ near the outflow boundary at $x = 0$ and to two parabolic layers of width $O(\sqrt{\varepsilon})$ near the characteristic boundaries at $y = 0$ and $y = 1$; see Fig. 1.

Furthermore we shall assume that

$$c + \frac{1}{2}b_x \geq \gamma > 0$$
which ensures the coercivity of the bilinear form associated with the differential operator $L$. Therefore (1.1) possesses a unique solution in $H^1_0(\Omega)$. Note that (1.2) can always be ensured by a simple transformation $\tilde{u}(x, y) = u(x, y)e^{\kappa x}$ with $\kappa$ chosen appropriately. Because of the presence of layers the use of quasi uniform meshes does not give accurate approximations of (1.1) unless the mesh size is of the order of the perturbation parameter which in practice constitutes a prohibitive restriction. Therefore layer-adapted meshes have to be used to obtain efficient discretizations. Based on a priori knowledge of the layer behaviour we shall construct piecewise uniform meshes—so called Shishkin meshes—that resolve the layers and yield robust (or uniform) convergence.

On these meshes two finite element discretizations will be analysed: the standard Galerkin FEM and the streamline-diffusion FEM (SDFEM) first proposed by Hughes and Brooks [4] both using bilinear trial and test functions. For problems of type (1.1) with only exponential layers both methods on Shishkin meshes are well understood. For the Galerkin method uniform convergence of almost first order in the energy norm was established by O’Riordan and Stynes [10], while Zhang [13] and Linß [7] proved uniform superconvergence of almost second order in discrete versions of that norm. The SDFEM was studied by Stynes and Tobiska [11] who prove uniform superconvergence in the streamline-diffusion norm of almost second order.

In the present paper problems with parabolic (or characteristic) layers will be considered. Practically these are more important. They can be considered as models of the flow past a surface. Particular attention will be paid to the choice of the streamline-diffusion parameter inside the parabolic layers where the mesh is aligned to the flow and anisotropically
refined. It was observed [5, 8] that when the stabilization parameter is chosen according to standard recommendations [12, p. 233] proportional to the streamline diameter of the mesh cell the accuracy is adversely affected. An alternative—and significantly smaller—choice based on residual free bubbles is advertised in [8]. However the argument is not rigorous and used some heuristic arguments.

Here we shall give for the first time a rigorous analysis of the Galerkin FEM and the SDFEM on Shishkin meshes for convection-diffusion problem with parabolic layers. For both methods uniform superconvergence of almost second order will be established. Simultaneously the issue of choosing the stabilization parameter in the SDFEM is addressed. The optimal choice inside the parabolic layers will turn out to be of order $\varepsilon^{-1/4} h_{sd}^2$, where $h_{sd}$ is the streamline diameter of the relevant mesh cells.

The paper is organized as follows. In first Section properly adapted Shishkin meshes are constructed based on knowledge of the layer behaviour of the solution. Section 3 gives bounds for the interpolation error which are then used in Sections 4 and 5 to study the convergence and superconvergence properties of the Galerkin FEM and SDFEM. Numerical experiments that illustrate our theoretical results are presented in Section 6. We remark that the analysis is full of a significant number of very technical details. In most cases these are deferred to appendices. In doing so we are attempting to provide a presentation of the essential ideas and results that is easier to access.

**Notation.** Throughout $C$ denotes a generic constant that is independent of both the perturbation parameter $\varepsilon$ and the number of mesh points used.

## 2 Solution decomposition and layer-adapted meshes

As mentioned before the solution $u$ of (1.1) has an exponential layer at $x = 0$ and two parabolic layers at $y = 0$ and $y = 1$. For our later analysis we shall suppose that $u$ can be split into a regular solution component and various layer parts:

**Assumption 2.1.** The solution $u$ of (1.1) can be decomposed as

$$u = v + w_1 + w_2 + w_{12},$$

where for all $x, y \in [0, 1]$ and $0 \leq i + j \leq 2$ we have the pointwise estimates

$$\left| \partial_x^i \partial_y^j v(x, y) \right| \leq C, \quad \left| \partial_x^i \partial_y^j w_1(x, y) \right| \leq C\varepsilon^{-i} e^{-\beta x/\varepsilon}, \quad (2.1a)$$

$$\left| \partial_x^i \partial_y^j w_2(x, y) \right| \leq C\varepsilon^{-j/2} \left( e^{-y/\sqrt{\varepsilon}} + e^{-(1-y)/\sqrt{\varepsilon}} \right), \quad (2.1b)$$

$$\left| \partial_x^i \partial_y^j w_{12}(x, y) \right| \leq C\varepsilon^{-(i+j/2)} e^{-\beta x/\varepsilon} \left( e^{-y/\sqrt{\varepsilon}} + e^{-(1-y)/\sqrt{\varepsilon}} \right) \quad (2.1c)$$

and for $0 \leq i + j \leq 3$ the $L_2$ bounds

$$\left\| \partial_x^i \partial_y^j v \right\|_{0, \Omega} \leq C, \quad \left\| \partial_x^i \partial_y^j w_1 \right\|_{0, \Omega} \leq C\varepsilon^{-i+1/2}, \quad (2.2a)$$

$$\left\| \partial_x^i \partial_y^j w_2 \right\|_{0, \Omega} \leq C\varepsilon^{-j/2+1/4}, \quad \left\| \partial_x^i \partial_y^j w_{12} \right\|_{0, \Omega} \leq C\varepsilon^{-i-j/2+3/4} \quad (2.2b)$$
and
\[ \left\| \partial_i^p \partial_j^q u \right\|_{0,\Omega} \leq C \left( 1 + \varepsilon^{-i+1/2} + \varepsilon^{-j/2+1/4} + \varepsilon^{-i-j/2+3/4} \right). \] (2.2c)

Remark 2.2. For \( i+j \leq 2 \) the \( L_2 \) bounds (2.2) follow from from the pointwise bounds (2.1).

When discretizing (1.1), we use a piecewise uniform mesh—a so-called *Shishkin mesh*—with \( N \) mesh intervals in both \( x \)- and \( y \)-direction which condenses in the layer regions. For this purpose define the mesh transition parameters
\[
\lambda_x := \min \left\{ \frac{1}{2}, \frac{\sigma \varepsilon}{\beta} \ln N \right\} \quad \text{and} \quad \lambda_y := \min \left\{ \frac{1}{4}, \sigma \sqrt{\varepsilon} \ln N \right\}
\]
with some positive parameter \( \sigma \) that will be fixed later.

The domain \( \Omega \) is divided into four (six) subregions—see Fig. 2—with \( \Omega_{12} \) covering the exponential layer, \( \Omega_{21} \) the parabolic layer and \( \Omega_{22} \) the corner layer and \( \Omega_{11} \) the remaining region which does not have layers. These subdomains will be uniformly dissected to give

\[
\begin{array}{c|c}
\Omega_{22} & \Omega_{21} \\
\hline
\Omega_{12} & \Omega_{11} \\
\hline
\Omega_{22} & \Omega_{21}
\end{array}
\]

*Figure 2: Dissection of \( \Omega \)*

For the mere sake of simplicity in our subsequent analysis we shall assume that
\[
\lambda_x = \frac{\sigma \varepsilon}{\beta} \ln N \leq \frac{1}{2} \quad \text{and} \quad \lambda_y = \sigma \sqrt{\varepsilon} \ln N \leq \frac{1}{4} \quad (2.3)
\]
as is typically the case for (1.1).

Note the mesh transition parameters \( \lambda_x \) and \( \lambda_y \) have been chosen such that the layer terms of \( u \) are of order \( N^{-\sigma} \) on \( \Omega_{11} \), i.e.,
\[
|w_1(x, y)| + |w_2(x, y)| + |w_{12}(x, y)| \leq CN^{-\sigma} \quad \text{for} \ (x, y) \in \Omega_{11}.
\]

Typically \( \sigma \) is chosen equal to the formal order of the method or to accommodate the error analysis: We shall require \( \sigma \geq 5/2 \) for technical reasons.

In order to construct our final mesh let \( N \) be divisible by 4. Divide each of the intervals \([0, \lambda_x]\) and \([\lambda_x, 1]\) uniformly into \( N/2 \) subintervals. We get our mesh in \( x \)-direction: \( x_i \),
The mesh is \( y \)-direction, \( y_j, j = 0, \ldots, N \), is obtained by uniformly dividing \([0, \lambda_y]\) and \([1 - \lambda_y, 1]\) into \( N/4 \) subintervals and \([\lambda_y, 1 - \lambda_y]\) into \( N/2 \) subintervals. By drawing lines through these mesh points parallel to the \( x \)-axis and \( y \)-axis the domain \( \Omega \) is partitioned into rectangles. This triangulation is denoted by \( T^N \); see Fig. 3.

\[
\begin{align*}
\text{Figure 3: Triangulation } T^N \text{ of } \Omega
\end{align*}
\]

The mesh sizes \( h_i := x_i - x_{i-1} \) and \( k_j := y_j - y_{j-1} \) satisfy

\[
\begin{align*}
h_i = \begin{cases} 
  h_2 := \frac{2\sigma \varepsilon \ln N}{\beta N}, & \text{for } i = 1, \ldots, N/2, \\
  h_1 := \frac{2(1 - \lambda_x)}{N}, & \text{for } i = N/2 + 1, \ldots, N
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
k_j = \begin{cases} 
  k_2 := \frac{4\sigma \sqrt{\varepsilon} \ln N}{N}, & \text{for } j = 1, \ldots, N/4 \text{ and } j = 3N/4 + 1, \ldots, N \\
  k_1 := \frac{2(1 - 2\lambda_y)}{N}, & \text{for } i = N/4 + 1, \ldots, 3N/4.
\end{cases}
\end{align*}
\]

Note the mesh sizes \( h \) and \( k \) satisfy

\[
h_2 \leq N^{-1} \leq h_1 \leq 2N^{-1} \text{ and } k_2 \leq N^{-1} \leq k_1 \leq 2N^{-1}, \quad (2.4)
\]

properties which are essential when inverse inequalities are applied in our later analysis. For the mesh elements we shall use two notations: \( \tau_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \) for a specific element, and \( \tau \) for a generic mesh rectangle with base \( h \) and height \( k \).

3 The interpolation error

Problem (1.1) will be discretized by means of finite element methods. Their analysis requires precise knowledge of the interpolation error which will be provided in this section.
Given any \( u \in C^0(\overline{\Omega}) \) and a triangulation \( T^N \) of \( \Omega \) into rectangles we denote by \( u^I \) the nodal piecewise bilinear interpolant to \( u \) over \( T^N \).

The main interpolation-error results can be obtained using the technique in [3, 10]. The adaptation is a straightforward, though tedious task.

**Theorem 3.1.** Let \( u^I \) be the linear or bilinear interpolant of \( u \) on a Shishkin mesh with \( \sigma \geq 2 \). Then the interpolation error satisfies

\[
\| u - u^I \|_{L^\infty(\Omega_{11})} \leq C N^{-2}, \quad \| u - u^I \|_{L^\infty(\Omega \setminus \Omega_{11})} \leq C N^{-2} \ln^2 N
\]

and

\[
\| u - u^I \|_\varepsilon \leq C N^{-1} \ln N.
\]

**Remark 3.2.** Bounds for the interpolation error in the \( L_2 \) norm on the various subdomains of \( \Omega \) are easily obtained from the \( L_\infty \) bounds:

\[
\| u - u^I \|_{0,\Omega_{k\ell}} \leq (\text{meas } \Omega_{k\ell})^{1/2} \| u - u^I \|_{\infty,\Omega_{k\ell}}, \quad k, \ell = 1, 2.
\]

The proof makes use of the following anisotropic interpolation error bounds given in [1, Lemma 7.11]

\[
\| w - w^I \|_{L_p(\tau)} \leq C \left\{ h^2 \| w_{xx} \|_{L_p(\tau)} + hk \| w_{xy} \|_{L_p(\tau)} + k^2 \| w_{yy} \|_{L_p(\tau)} \right\}
\]

and

\[
\| (w - w^I)_x \|_{L_p(\tau)} \leq C \left\{ h \| w_{xx} \|_{L_p(\tau)} + k \| w_{xy} \|_{L_p(\tau)} \right\}
\]

which hold true for \( p \in [1, \infty] \) and arbitrary \( w \in W^2_p(\Omega) \). Furthermore

\[
\| (w - w^I)_x \|_{L_\infty(\tau)} \leq \| w_x \|_{L_\infty(\tau)} + \| w^I_x \|_{L_\infty(\tau)} \leq 2 \| w_x \|_{L_\infty(\tau)}
\]

is used. Clearly analogous results hold true for \( (w - w^I)_y \).

For our later superconvergence analysis we require more detailed results for the interpolation operator applied to the various parts of the solution decomposition. These are given in Appendix A.

## 4 Galerkin FEM

The variational formulation of (1.1) is: Find \( u \in H^1_0(\Omega) \) such that

\[
a_{\text{Gal}}(u, v) := \varepsilon (\nabla u, \nabla v) - (b u_x, v) + (c u, v) = f(v) := (f, v) \text{ for all } v \in H^1_0(\Omega),
\]

where \((\cdot, \cdot)_D\) denotes the standard scalar product in \( L_2(D) \). If \( D = \Omega \) we drop the \( \Omega \) from the notation.
The bilinear form $a_{\text{Gal}}(\cdot, \cdot)$ is coercive with respect to the $\varepsilon$-weighted energy norm, i.e.,

$$a_{\text{Gal}}(v, v) \geq \varepsilon |v|_1^2 + \gamma \|v\|_0^2 =: \|v\|_\varepsilon^2 \quad \text{for all } v \in H^1_0(\Omega),$$

where $| \cdot |_{1,D}$ is the standard seminorm in $H^1_1(D)$-seminorm and $\| \cdot \|_{0,D}$ that in $L^2_2(D)$. Again the $D$ is dropped if $D = \Omega$. Because of the coercivity the Lax-Milgram Lemma ensures the existence of a unique solution $u \in H^1_0(\Omega)$ of the variational formulation. Let $V^N \subset H^1_0(\Omega)$ be a finite-element space consisting of piecewise bilinear element over the Shishkin mesh. Then the discretisation is: Find $u^N \in V^N$ such that

$$a_{\text{Gal}}(u^N, v^N) = f(v^N) \quad \text{for all } v^N \in V^N. \quad (4.2)$$

The uniqueness of this solution is guaranteed by the coercivity of $a_{\text{Gal}}$.

### 4.1 Convergence

In this section we establish a first convergence result for the Galerkin FEM on Shishkin meshes for (1.1). The technique is adapted from [10] Theorem 4.1.

**Theorem 4.1.** Let $\sigma \geq 2$. Then

$$\|u - u^N\|_\varepsilon \leq CN^{-1} \ln N.$$  

**Proof.** Let $\eta := u^I - u$ and $\chi := u^I - u^N$. Theorem 3.1 readily provides bounds for the interpolation error:

$$\|u - u^I\|_\varepsilon \leq CN^{-1} \ln N.$$  

When bounding $\chi$ we start from the coercivity and Galerkin-orthogonality of $a_{\text{Gal}}$:

$$\|\chi\|_\varepsilon^2 \leq a_{\text{Gal}}(\chi, \chi) = a_{\text{Gal}}(\eta, \chi) = \varepsilon (\nabla \eta, \nabla \chi) + (\eta, b_x \chi) + ((b_x + c)\eta, \chi) \leq C\|\eta\|_\varepsilon \|\chi\|_\varepsilon + C\left\{ \|\eta\|_{0,\Omega_{11}} \|\chi_x\|_{0,\Omega_{11}} + \|\eta\|_{0,\Omega_{21}} \|\chi_x\|_{0,\Omega_{21}} + \|\eta\|_{L^\infty(\Omega_{12} \cup \Omega_{22})} \|\chi_x\|_{L^1(\Omega_{12} \cup \Omega_{22})} \right\}. \quad (4.3)$$

The inverse inequality (B.1) gives

$$N^{-1} \|\chi_x\|_{0,\Omega_{11}} \leq C \|\chi\|_{0,\Omega_{11}} \leq C \|\chi\|_\varepsilon$$

and

$$N^{-1/2} \varepsilon^{1/4} \|\chi_x\|_{0,\Omega_{21}} = \left( N^{-1} \|\chi_x\|_{0,\Omega_{21}} \varepsilon^{1/2} \|\chi_x\|_{0,\Omega_{21}} \right)^{1/2} \leq C \left( \|\chi\|_{0,\Omega_{21}} \varepsilon^{1/2} \|\chi_x\|_{0,\Omega_{21}} \right)^{1/2} \leq C \|\chi\|_\varepsilon,$$
while on $\Omega_{12} \cup \Omega_{22}$ the Cauchy-Schwarz inequality yields

$$\|\chi\|_{L^1(\Omega_{12} \cup \Omega_{22})} \leq (\text{meas}(\Omega_{12} \cup \Omega_{22}))^{1/2} \|\chi\|_{0,\Omega_{12} \cup \Omega_{22}} \leq C\varepsilon^{1/2} N\|\chi\|_{\varepsilon}.$$ 

Apply these three bounds to (4.3) and recall the results of Theorem 3.1. We get

$$\|\|\chi\|_{\varepsilon} \leq C\left(N^{-1} \ln N + N^{-1} + N^{-3/2} \ln^{5/2} N + N^{-2} \ln^{5/2} N\right) \leq CN^{-1} \ln N.$$ 

Finally a triangle inequality completes the proof. 

**Remark 4.2.** The result of this Theorem also holds when linear or mixed linear/bilinear elements are used in the discretization.

### 4.2 Superconvergence

In the previous section we have established almost first order uniform convergence of the Galerkin discretization on Shishkin meshes. This result can be further improved to almost second order when the difference between numerical solution and the interpolant of the exact solution is considered.

**Theorem 4.3.** Let $\sigma \geq 5/2$. Then the Galerkin-FEM solution $u^N$ satisfies

$$\|\|u^I - u^N\|_{\varepsilon} \leq CN^{-2} \ln^2 N$$

**Remark 4.4.** A similar result was established in [7] for problems with exponential layers only and later slightly improved upon by the author [9]. However, due to the different nature of the parabolic layers, the term $w_2$ in the decomposition, the analysis from the afore mentioned papers requires a number of changes and additional new ideas.

The proof of Theorem 4.3 starts from the coercivity and Galerkin orthogonality of the bilinear form $a_{Gal}(\cdot,\cdot)$.

$$\|\|u^I - u^N\|_{\varepsilon}^2 \leq |a_{Gal}(u - u^I, u^I - u^N)|$$

$$\leq \varepsilon \left| \left( \nabla (u - u^I), \nabla (u^I - u^N) \right) \right|$$

$$+ \left| \left( b(u - u^I)_x, u^I - u^N \right) \right| + \left| \left( c(u - u^I), u^I - u^N \right) \right|.$$ 

In Appendix C it will be shown that all three terms on the right-hand side can be bounded by $CN^{-2} \ln^2 N\|u^I - u^N\|_{\varepsilon}$. The statement of the Theorem follows upon dividing by $\|\|u^I - u^N\|_{\varepsilon}$.

The crucial ingredients in the analysis are the following integral identities from [6].

**Lemma 4.5.** Let $\tau_{ij} \in T^N$ be an arbitrary mesh rectangle with midpoint $(\bar{x}_i, \bar{y}_j)$ and edges $\ell_1, \ell_2$ that are parallel to the y-axis.
For any function \( w \in C^3(\bar{\tau}_{ij}) \) and any bilinear function \( \chi \) there holds
\[
\int_{\tau_{ij}} (w - w^T)_x \chi_x = \int_{\tau_{ij}} \left[ F_j \chi_x - \frac{1}{3} (F_j^2)^\prime \chi_{xy} \right] w_{xyy} \tag{4.4}
\]
and
\[
\int_{\tau_{ij}} (w - w^T)_x \chi = H_{ij}(w, \chi) + \frac{h_i^2}{12} \left( \int_{t_1} - \int_{t_2} \right) \chi w_{xx} dy \tag{4.5}
\]
with
\[
H_{ij}(w, \chi) := \int_{\tau_{ij}} \left[ F_j (\chi - E^i_x \chi_x) - \frac{(F_j^2)^\prime}{3} (\chi_y - E^i_x \chi_{xy}) \right] w_{xyy} + \int_{\tau_{ij}} \left[ \frac{(E_j^2)^\prime}{6} \chi_x - \frac{h_i^2}{12} \chi \right] w_{xxx}
\]
and
\[
E_i(x) := \frac{(x - \tilde{x}_i)^2}{2} - \frac{h_i^2}{8} \quad \text{and} \quad F_j(y) := \frac{(y - \tilde{y}_j)^2}{2} - \frac{k_j^2}{8}.
\]

**Remark 4.6.** The Cauchy-Schwarz inequality and the inverse inequality (B.1) applied to (4.4) give
\[
\left| \left( (w - w^T)_x, \chi_x \right)_{\tau_{ij}} \right| \leq C k_j^2 \| w_{xyy} \|_{0, \tau_{ij}} \| \chi_x \|_{0, \tau_{ij}} \quad \text{for all } w \in C^3(\bar{\tau}_{ij}); \tag{4.6}
\]
similarly
\[
|H_{ij}(w, \chi)| \leq C \left\{ k_j^2 \| w_{xyy} \|_{0, \tau_{ij}} + h_i^2 \| w_{xxx} \|_{0, \tau_{ij}} \right\} \| \chi \|_{0, \tau_{ij}}. \tag{4.7}
\]

Thus both terms are formally of second order. However in the context of the present paper the uniformity with respect to \( \varepsilon \) remains to be established. Furthermore, the line integrals in (4.5) do not cancel on a strongly non-uniform mesh like the Shishkin mesh we are using and therefore require careful treatment; see Appendix C.

## 5 Streamline-diffusion FEM

The streamline-diffusion FEM adds weighted residuals to the standard Galerkin FEM in order to stabilize the discretization:
\[
a(u, v) + \sum_{\tau \in T^N} \delta_{\tau} (f - Lu, bv_x)_x = f(v), \tag{5.1}
\]
where \( \delta \geq 0 \) is a user chosen parameter. This modification is consistent with (1.1), i.e., its solution solves (5.1) too.
Our discretization reads: Find \( u^N \in V^N \) such that
\[
a_{SD}(u^N, v^N) := a_{Gal}(u^N, v^N) + a_{stab}(u^N, v^N) = f_{SD}(v^N) \quad \text{for all} \ v^N \in V^N,
\]
with
\[
a_{stab}(u, v) := \sum_{\tau \in T^N} \delta_{\tau}((bu_x, bv_x)_{\tau} - (cu, bv_x)_{\tau})
\]
and
\[
f_{SD}(u, v) := f(v) - \sum_{\tau \in T^N} \delta_{\tau}(f, bv_x)_{\tau}.
\]
The SDFEM satisfies the Galerkin orthogonality
\[
a_{SD}(u - u^N, v^N) = 0 \quad \text{for all} \ v \in V^N
\]
and is coercive with respect to the streamline diffusion norm
\[
\|v\|_{SD}^2 := \|v\|^2 + \sum_{\tau \in T^N} \delta_{\tau}(bv_x, bv_x)_{\tau}.
\]
It is shown in, e.g., [12, §III.3.2.1] that if
\[
0 \leq \delta_{\tau} \quad \text{and} \quad \delta_{\tau}\|e\|_{L^\infty(\tau)}^2 \leq \gamma \quad \text{for all} \ \tau \in T^N
\]
then
\[
a_{SD}(v, v) \geq \frac{1}{2}\|v\|_{SD}^2 \quad \forall v \in V^N.
\]
We remark that \( \|v\|_e \leq C\|v\|_{SD} \) for all \( v \in H^1_0(\Omega) \). Thus \( a_{SD}(\cdot, \cdot) \) enjoys a stronger stability than \( a_{Gal}(\cdot, \cdot) \). Roughly speaking, we can say that the larger \( \delta \) the more stability is introduced into the method.

Our error analysis starts again from the coercivity and Galerkin orthogonality:
\[
\frac{1}{2}\|u^I - u^N\|_{SD}^2 \leq a_{Gal}(u^I - u, u^I - u^N) + a_{stab}(u^I - u, u^I - u^N).
\]
For the first term on the right-hand side we have on Shishkin meshes with \( \sigma \geq 5/2 \)
\[
|a_{Gal}(u^I - u, \chi)| \leq CN^{-2}N\|\chi\|_e \leq CN^{-2}N\|\chi\|_{SD} \quad \text{for all} \ \chi \in V^N;
\]
see proof of Theorem 4.3. It remains to bound the second term in (5.3) and to analyse the influence of the parameter \( \delta \) on stability and accuracy. This will be done in the next section.
5.1 Superconvergence and choice of $\delta$

The streamline diffusion parameter $\delta$ is chosen to be constant on each subdomain of the decomposition of $\Omega$, i.e., we set

$$\delta|_\tau = \delta_\tau = \delta_{k\ell} \quad \text{if } \tau \subset \Omega_{k\ell}, \quad k, \ell = 1, 2.$$ 

**Theorem 5.1.** Let $u^N$ be the streamline-diffusion approximation to $u$ on a family of Shishkin meshes with $\sigma \geq 5/2$. Suppose the stabilization parameter $\delta$ satisfies (5.2),

$$\delta_{12} \leq C\varepsilon N^{-2}, \quad \delta_{21} \leq C\varepsilon^{-1/4} N^{-2}, \quad \delta_{22} \leq C\varepsilon^{3/4} N^{-2}$$

and

$$\delta_{11} \leq \begin{cases} 
C\varepsilon N^{-1} & \text{if } \varepsilon \leq N^{-1}, \\
C\varepsilon^{-1} N^{-2} & \text{if } \varepsilon \geq N^{-1}.
\end{cases}$$

with some positive constant $C^*$ independent of $\varepsilon$ and the mesh. Then

$$\left\| u^I - u^N \right\|_{SD} \leq C N^{-2} \ln^2 N.$$

**Proof.** Set $\chi = u^I - u^N$ and $\eta = u^I - u$. Then by (5.3) and (5.4), we have

$$\frac{1}{2} \left\| \chi \right\|_{SD}^2 \leq a_{stab}(\eta, \chi) + C N^{-2} \ln^2 N \left\| \chi \right\|_{SD}.$$ 

On the four subdomains of $\Omega$ we have the bounds

$$|a_{stab}(\eta, \chi)_{\Omega_{11}}| \leq C \left( \varepsilon \delta_{11} \ln^{1/2} N + \delta_{11}^{1/2} N^{-\sigma+1} \right) \left\| \chi \right\|_{SD},$$

$$|a_{stab}(\eta, \chi)_{\Omega_{21}}| \leq C \varepsilon^{1/4} \left( \delta_{21} + \delta_{21}^{1/2} N^{-\sigma+1} \right) \ln^{1/2} N \left\| \chi \right\|_{SD},$$

$$|a_{stab}(\eta, \chi)_{\Omega_{12}}| \leq C \left( \delta_{12}^{1/2} \varepsilon^{-1/2} N^{-\sigma+1} \ln^{-1/2} N + \delta_{12} \varepsilon^{-1} \right) \left\| \chi \right\|_{SD},$$

and

$$|a_{stab}(\eta, \chi)_{\Omega_{22}}| \leq C \delta_{22} \varepsilon^{-3/4} \ln N \left\| \chi \right\|_{SD}.$$ 

The proof of Lemma 4.4 in [11] requires only minor modifications to get (5.5a). Details for the other three estimates can be found in Appendix D.

Note that our choice of $\delta_{11}$ implies $\varepsilon \delta_{11} \leq C N^{-2}$ and $\delta_{11} \leq C N^{-1}$. The proposition of the theorem follows. \qed

**Remark 5.2.** Theorems 5.1 and 3.1, and the triangle inequality provide bounds for the error in the $\varepsilon$-weighted energy norm:

$$\left\| u - u^N \right\|_{\varepsilon} \leq C N^{-1} \ln N.$$
Table 1: Problem I – Galerkin FEM

6 Numerical Results

Two different test problems will be studied. In our experiments the uniform errors are estimated by taking the maximum over various values of $\varepsilon$, i.e.

$$e^N_\ast = \max_{\varepsilon = 10^{-2}, 10^{-4}, ... , 10^{-10}} \|u - u^N\|_\ast.$$

The rate of convergence is estimated using the formula

$$r^N_\ast = \log \left( \frac{e^N_\ast}{e^{2N}_\ast} \right).$$

For the computations with the SDFEM we have chosen $\delta$ to be the maximal values allowed by Theorem 5.1 with $C^\ast = 1$.

**Problem I** is given by

$$-\varepsilon \Delta u - (2-x)u_x + \frac{3}{2} u = f \quad \text{in } \Omega = (0,1)^2, \quad u|_{\partial \Omega} = 0$$

with homogeneous Dirichlet boundary conditions and right-hand side $f$ chosen such that

$$u(x,y) = \left( \cos \frac{\pi x}{2} - \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} \right) \left( 1 - e^{-y/\sqrt{\varepsilon}} \right) \left( 1 - e^{-(1-y)/\sqrt{\varepsilon}} \right) \frac{1 - e^{-1/\sqrt{\varepsilon}}}{1 - e^{-1/\sqrt{\varepsilon}}}$$

is the exact solution. Tables 1 and 2 display the errors in various norms of the Galerkin FEM and the SDFEM for test problems I. They are clear illustrations of the first order convergence (Theorem 4.1, Remark 5.2) and the almost almost second order superconvergence result (Theorems 4.3, 5.1). The last columns of both tables give error estimates in the $L^\infty$ norm. For both methods we observe almost second order convergence, although we do not have theoretical justification for this behaviour.

In Tables 3 and 4 we verify the dependence of the errors on the perturbation parameter. We have fixed $N = 256$ and varied the perturbation parameter. For both methods we observe $\varepsilon$-independence of the errors in the energy norm, and for the SDFEM also in
the streamline-diffusion norm. However for the SDFEM we notice an increase of the $L_\infty$ error when $\varepsilon$ gets smaller. This may be an indication that the stabilization according to Theorem 5.1—though correct convergence in the SD norm—is not optimal for the maximum-norm error.

Problem II.

$$-\varepsilon \Delta u - u_x + u = f \quad \text{in} \ \Omega = (0,1)^2, \quad u|_{\partial \Omega} = 0$$

with right-hand side $f$ chosen such that

$$u = \sin(\pi x) \frac{(1 - e^{-y/\sqrt{\varepsilon}}) (1 - e^{-(1-y)/\sqrt{\varepsilon}})}{1 - e^{-1/\sqrt{\varepsilon}}}$$

is the solution. This problem has been selected because it exhibits only parabolic layers the layer we are particularly concerned with. The construction of the Shishkin mesh is changed by using a uniform mesh in $x$-direction since there is no layer at $x = 0$.

Tables 5 and 6 give the errors of the two methods. The behaviour is similar to Problem I: almost first order convergence and second order superconvergence in the energy norms.
\[
\begin{align*}
\varepsilon & \quad ||u - u^N||_\varepsilon \quad ||u^I - u^N||_\varepsilon \quad ||u^I - u^N||_{SD} \quad ||u - u^N||_{L_\infty} \\
1.00e-02 & \quad 1.8662e-02 \quad 1.4154e-04 \quad 1.4154e-04 \quad 1.3793e-03 \\
1.00e-03 & \quad 2.1488e-02 \quad 1.9974e-04 \quad 1.9974e-04 \quad 1.7082e-03 \\
1.00e-04 & \quad 2.2200e-02 \quad 2.5950e-04 \quad 2.6428e-04 \quad 4.8801e-03 \\
1.00e-05 & \quad 2.2125e-02 \quad 1.9751e-04 \quad 2.0525e-04 \quad 4.8892e-03 \\
1.00e-06 & \quad 2.2101e-02 \quad 1.7591e-04 \quad 1.8521e-04 \quad 4.9062e-03 \\
1.00e-07 & \quad 2.2094e-02 \quad 1.6992e-04 \quad 1.7988e-04 \quad 4.9412e-03 \\
1.00e-08 & \quad 2.2091e-02 \quad 1.6874e-04 \quad 1.7899e-04 \quad 5.0288e-03 \\
1.00e-09 & \quad 2.2091e-02 \quad 1.6876e-04 \quad 1.7915e-04 \quad 5.4620e-03 \\
1.00e-10 & \quad 2.2091e-02 \quad 1.6894e-04 \quad 1.7945e-04 \quad 5.4941e-03 \\
\end{align*}
\]

Table 4: Problem I – uniformity of the SDFEM

\[
\begin{align*}
N & \quad ||u - u^N||_\varepsilon \quad \text{rate} \quad ||u^I - u^N||_\varepsilon \quad \text{rate} \quad ||u - u^N||_{L_\infty} \quad \text{rate} \\
32 & \quad 2.127e-02 \quad 0.70 \quad 3.665e-03 \quad 1.42 \quad 7.871e-02 \quad 1.15 \\
64 & \quad 1.307e-02 \quad 0.76 \quad 1.368e-03 \quad 1.54 \quad 3.558e-02 \quad 1.34 \\
128 & \quad 7.700e-03 \quad 0.80 \quad 4.711e-04 \quad 1.61 \quad 1.409e-02 \quad 1.48 \\
256 & \quad 4.415e-03 \quad 0.83 \quad 1.544e-04 \quad 1.66 \quad 5.069e-03 \quad 1.58 \\
512 & \quad 2.486e-03 \quad 0.85 \quad 4.888e-05 \quad 1.70 \quad 1.698e-03 \quad 1.65 \\
1024 & \quad 1.382e-03 \quad 1.508e-05 \quad 5.426e-04 \quad 1.70 \\
\end{align*}
\]

Table 5: Problem II – Galerkin FEM

However the dependence of the errors on \(\varepsilon\) is different. For the energy norms we witness a decrease of the errors when \(\varepsilon\) gets smaller, while the maximum-norm error increase slightly.

References


Table 6: Problem II – SDFEM

<table>
<thead>
<tr>
<th>( N )</th>
<th>( |u - u^N|_\varepsilon ) rate</th>
<th>( |u' - u^N|_\varepsilon ) rate</th>
<th>( |u' - u^N|_{SD} ) rate</th>
<th>( |u - u^N|_{L^\infty} ) rate</th>
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<tr>
<td>32</td>
<td>2.126e-02 0.70</td>
<td>3.291e-03 1.39</td>
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Table 7: Problem II – uniformity of the Galerkin FEM

<table>
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<tr>
<th>( \varepsilon )</th>
<th>( |u - u^N|_\varepsilon )</th>
<th>( |u' - u^N|_\varepsilon )</th>
<th>( |u - u^N|_{L^\infty} )</th>
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</tbody>
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In this first appendix we derive interpolation error bounds for the various parts of the decomposition \( u = v + w_1 + w_2 + w_{12} \) that will be used later when giving details of the superconvergence analysis.

First, (3.2) and (2.2) give
\[
\| (v - v^I)_x \|_0 + \| (v - v^I)_y \|_0 \leq CN^{-1}. \tag{A.1}
\]
Next, use (3.3) and (2.1) to obtain
\[
\| (w_2 - w_2^I)_x \|_{L_\infty(\Omega_{11} \cup \Omega_{12})} + \varepsilon \| (w_{12} - w_{12}^I)_x \|_{L_\infty(\Omega_{12})} \leq CN^{-\sigma} \tag{A.2a}
\]
and
\[
\varepsilon \| (w_{12} - w_{12}^I)_x \|_{L_\infty(\Omega_{11})} \leq CN^{-2\sigma}. \tag{A.2b}
\]
Finally, bounds for the interpolation error of the layer terms are needed.

### Table 8: Problem II – uniformity of the SDFEM

<table>
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<tr>
<th>( \varepsilon )</th>
<th>( | u - u^N |_\varepsilon )</th>
<th>( | u^I - u^N |_\varepsilon )</th>
<th>( | u^I - u^N |_{SD} )</th>
<th>( | u - u^N |<em>{L</em>\infty} )</th>
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Proposition A.1. Let \( w = w_1 + w_2 + w_{12} \). Then

\[
\begin{align*}
\|w - w^f\|_{0,\Omega_{11}} &\leq C \left( \varepsilon^{1/4} N^{-\sigma} + N^{-\sigma - 1/2} \right) \quad \text{(A.3a)} \\
\|w - w^f\|_{0,\Omega_{12}} &\leq C \varepsilon^{1/2} \left( N^{-2} \ln^2 N + N^{-\sigma} \ln^{1/2} N \right) \quad \text{(A.3b)} \\
\|w - w^f\|_{0,\Omega_{21}} &\leq C \varepsilon^{1/4} \left( N^{-2} \ln^2 N + N^{-\sigma} \ln^{1/2} N \right) \quad \text{(A.3c)} \\
\|w - w^f\|_{0,\Omega_{22}} &\leq C \varepsilon^{1/2} N^{-2} \ln^2 N \quad \text{(A.3d)}
\end{align*}
\]

Proof. (i) Let us first consider \( \|w - w^f\|_{0,\Omega_{11}} \). Clearly

\[
\|w - w^f\|_{0,\Omega_{11}} \leq \|w\|_{0,\Omega_{11}} + \|w^f\|_{0,\Omega_{11}}.
\]

A direct calculation and (2.1) give

\[
\|w\|_{0,\Omega_{11}} \leq C \varepsilon^{1/4} N^{-\sigma}.
\]

When bounding \( w^f \) we follow an idea by Zhang [13]. This enables us to assume \( \sigma \geq 5/2 \) rather than \( \sigma \geq 3 \) in our later analysis.

The domain \( \Omega_{11} \) is divided into \( S = [\lambda_x + h_1, 1] \times [\lambda_y + k_1, 1 - \lambda_y - k_1] \) and \( \Omega_{11} \setminus S \).

Note that \( \Omega_{11} \setminus S \) consists of only a single ply of \( \mathcal{O}(N) \) mesh elements adjacent to the boundary layer regions. Thus

\[
\|w^f\|^2_{0,\Omega_{11}\setminus S} \leq \sum_{\tau \in \Omega_{11}\setminus S} h_1 k_1 \|w^f\|^2_{L_\infty(\tau)} \leq CN^{-1} \|w\|^2_{L_\infty(\Omega_{11})} \leq CN^{-2\sigma - 1}, \quad \text{(A.4)}
\]

by (2.4) and (2.1).

For \( \tau_{ij} \in S \) we have

\[
\begin{align*}
\|w^f\|^2_{0,\tau_{ij}} &\leq h_i k_j \|w^f\|^2_{L_\infty(\tau_{ij})} \\
&\leq C h_i k_j \left( e^{-2\beta x_{i-1}/\varepsilon} + \left( e^{-2y_j - 1/\sqrt{\varepsilon}} + e^{-2(1-y_j)/\sqrt{\varepsilon}} \right) \left( 1 + e^{-2\beta x_{i-1}/\varepsilon} \right) \right) \\
&\leq C \left\{ \int_{\tau_{i-1,j-1}} e^{-2\beta x/\varepsilon} + e^{-2y/\sqrt{\varepsilon}} \left( 1 + e^{-2\beta x/\varepsilon} \right) \right. \\
&\quad + \left. \int_{\tau_{i-1,j}} e^{-2(1-y)/\sqrt{\varepsilon}} \left( 1 + e^{-2\beta x/\varepsilon} \right) \right\}
\end{align*}
\]

Summing over \( S \), we obtain

\[
\|w^f\|^2_{0,S} \leq C \int_{\Omega_{11}} e^{-2\beta x/\varepsilon} + \left( e^{-2y/\sqrt{\varepsilon}} + e^{-2(1-y)/\sqrt{\varepsilon}} \right) \left( 1 + e^{-2\beta x/\varepsilon} \right) \leq C \varepsilon^{1/2} N^{-2\sigma},
\]

which together with (A.4) completes the proof of (A.3a).

(ii) On \( \Omega_{12} \) use (3.1) and (2.2) to establish

\[
\|w_1 - w_{11}^f\|_{0,\Omega_{12}} \leq C \varepsilon^{1/2} N^{-2} \ln^2 N,
\]
while for \( w_2 \) and \( w_{12} \) we proceed as follows

\[
\left\| w_2 + w_{12} - (w_2 + w_{12})^I \right\|_{0, \Omega_{12}} \\
\leq C (\text{meas } \Omega_{12})^{1/2} \left\| w_2 + w_{12} \right\|_{L_\infty(\Omega_{12})} \leq C \varepsilon^{1/2} N^{-\sigma} \ln^{1/2} N.
\]

We get (A.3b).

(iii) On \( \Omega_{21} \) we employ (3.1) to bound the error in \( w_2 \):

\[
\left\| w_2 - w_2^I \right\|_{0, \Omega_{21}} \leq C \varepsilon^{1/4} N^{-2} \ln^2 N.
\]

On the other hand

\[
\left\| w_1 + w_{12} - (w_1 + w_{12})^I \right\|_{0, \Omega_{21}} \\
\leq (\text{meas } \Omega_{21})^{1/2} \left\| w_1 + w_{12} \right\|_{L_\infty(\Omega_{21})} \leq C \varepsilon^{1/4} N^{-\sigma} \ln^{1/2} N.
\]

This is (A.3c).

(iv) Finally, we study \( w - w^I \) on \( \Omega_{22} \). Ineq. (3.1) gives

\[
\left\| w - w^I \right\|_{0, \Omega_{22}} \leq CN^{-2} \ln^2 N \left\{ \varepsilon^2 \left\| w_{x x} \right\|_{0, \Omega_{22}} + \varepsilon^{3/2} \left\| w_{x y} \right\|_{0, \Omega_{22}} + \varepsilon \left\| w_{y y} \right\|_{0, \Omega_{22}} \right\}
\]

\[
\leq C \varepsilon^{3/4} N^{-2} \ln^{5/2},
\]

by (2.1) and a direct calculation. Using (2.3), we obtain (A.3d).

\[\square\]

### B Inverse estimates

Throughout the remaining analysis we shall make frequent use of the following inverse estimates. Let \( \chi \) be a polynomial on the mesh rectangle \( \tau \). Then

\[
\left\| \chi_x \right\|_{L_p(\tau)} \leq Ch^{-1} \left\| \chi \right\|_{L_p(\tau)} \quad \text{and} \quad \left\| \chi_y \right\|_{L_p(\tau)} \leq Ck^{-1} \left\| \chi \right\|_{L_p(\tau)} \quad \text{(B.1)}
\]

\[
\int_{y_{j-1}}^{y_j} |\chi(x_i, y)| \, dy \leq Ch_i^{-1} \left\| \chi \right\|_{L_1(\tau_{i j})} \quad \text{(B.2)}
\]

\[
\left\| \chi \right\|_{L_q(\tau)} \leq C(\text{meas } \tau)^{1/q - 1/p} \left\| \chi \right\|_{L_p(\tau)} \quad \text{for } p, q \in [1, \infty] \quad \text{(B.3)}
\]

where \( h \) is the base and \( k \) the height of \( \tau \).

### C Proof of Theorem 4.3

**Proposition C.1.** Let \( \sigma \geq 2 \). Then the solution \( u^N \) of the Galerkin discretization (4.2) satisfies

\[
\varepsilon \left| \langle \nabla (u - u^I), \nabla \chi \rangle \right| \leq CN^{-2} \ln^2 N \left\| \chi \right\|_e \quad \text{for all } \chi \in V^N.
\]
Proof. Summing (4.6) over all elements in \( \Omega_{k\ell} \), one of our four subdomains of \( \Omega \), and using a discrete Cauchy-Schwarz inequality, we get

\[
\left| \left( (w - w^I)_x, \chi_x \right)_{\Omega_{k\ell}} \right| \leq Ck^2 \| w_{xy} \|_{0,\Omega_{k\ell}} \| \chi_x \|_{0,\Omega_{k\ell}} \text{ for } k, \ell = 1, 2 \tag{C.1}
\]

and —because of symmetry—

\[
\left| \left( (w - w^I)_y, \chi_y \right)_{\Omega_{k\ell}} \right| \leq Ck^2 \| w_{xy} \|_{0,\Omega_{k\ell}} \| \chi_y \|_{0,\Omega_{k\ell}} \text{ for } k, \ell = 1, 2. \tag{C.2}
\]

(i) First let us consider \( ((u - u^I)_x, \chi_x) \). Inequalities (C.1) and (2.2) yield

\[
\varepsilon \left| \left( (u - u^I)_x, \chi_x \right)_{\Omega_{22\cup\Omega_{21}}} \right| \leq C\varepsilon^{1/4} N^{-2} \ln^2 N \| \chi \|_\varepsilon \tag{C.3}
\]

and —recalling the decomposition \( u = v + w_1 + w_2 + w_{12} \)—

\[
\varepsilon \left| \left( ((v + w_1) - (v + w_1)^I)_x, \chi_x \right)_{\Omega_{12\cup\Omega_{11}}} \right| \leq CN^{-2} \| \chi \|_\varepsilon. \tag{C.4}
\]

because \( k_2 = 4\varepsilon^{1/2}\sigma N^{-1} \ln N \) and \( k_1 \leq 2N^{-1} \).

The Hölder and the Cauchy-Schwarz inequalities yield

\[
\left| \left( (w - w^I)_x, \chi_x \right)_D \right| \leq \| (w - w^I)_x \|_{L_\infty(D)} \| \chi_x \|_{L_1(D)} \leq \| (w - w^I)_x \|_{L_\infty(D)} \left( \text{meas}(D) \right)^{1/2} \| \chi_x \|_{0,D}.
\]

From this inequality and (A.2), we obtain

\[
\varepsilon \left| \left( (w_2 + w_{12}) - (w_2 + w_{12})^I)_x, \chi_x \right)_{\Omega_{12}} \right| \leq CN^{-\sigma} \ln^{1/2} N \| \chi \|_\varepsilon \tag{C.5}
\]

and

\[
\varepsilon \left| \left( (w_2 - w_2^I)_x, \chi_x \right)_{\Omega_{11}} \right| \leq C\varepsilon^{1/2} N^{-\sigma} \| \chi \|_\varepsilon, \tag{C.6}
\]

because \( \text{meas}(\Omega_{12}) \leq \sigma\varepsilon\beta^{-1} \ln N \) and \( \text{meas}(\Omega_{11}) \leq 1 \).

Next consider the corner layer term \( w_{12} \) on \( \Omega_{11} \). Starting again from the Hölder inequality, we get

\[
\varepsilon \left| \left( (w_{12} - w_{12}^I)_x, \chi_x \right)_{\Omega_{11}} \right| \leq \varepsilon \| (w_{12} - w_{12}^I)_x \|_{L_\infty(\Omega_{11})} \| \chi_x \|_{0,\Omega_{11}} \leq CN^{-2\sigma+1} \| \chi \|_\varepsilon, \tag{C.7}
\]

by (A.2), the inverse inequality (B.1) and \( h_1 \geq N^{-1} \).

Collecting (C.3)–(C.7), we get

\[
\varepsilon \left| \left( (u - u^I)_x, \chi_x \right)_{\Omega} \right| \leq CN^{-2} \ln^2 N \| \chi \|_\varepsilon. \tag{C.8}
\]

(ii) In the second part of the proof we study \( ((u - u^I)_y, \chi_y) \).
Similar to the first part of the analysis, (C.2) and (2.2) give
\[
\varepsilon \left| \left( (u - u^I) y, \chi_y \right)_{\Omega_{22} \cup \Omega_{12}} \right| \leq C \varepsilon^{3/4} N^{-2} \ln^2 N \|\chi\| \varepsilon \tag{C.9}
\]
and
\[
\varepsilon \left| \left( ((v + w_2) - (v + w_2)^I) y, \chi_y \right)_{\Omega_{11} \cup \Omega_{21}} \right| \leq C \varepsilon^{1/4} N^{-2} \|\chi\| \varepsilon, \tag{C.10}
\]
because \( h_2 = 2\varepsilon \beta^{-1} N^{-1} \ln N \) and \( h_1 \leq 2N^{-1} \).

Imitate the argument that lead to (C.5) and (C.6) to obtain
\[
\varepsilon \left| \left( ((w_1 + w_{12}) - (w_1 + w_{12})^I) y, \chi_y \right)_{\Omega_{11}} \right| \leq C \varepsilon^{1/4} N^{-\sigma} \ln^{1/2} N \|\chi\| \varepsilon \tag{C.11}
\]
and
\[
\varepsilon \left| \left( (w_1 - w_1^I) y, \chi_y \right)_{\Omega_{11}} \right| \leq C \varepsilon^{1/2} N^{-\sigma} \|\chi\| \varepsilon, \tag{C.12}
\]

since \( \text{meas}(\Omega_{21}) \leq 2\varepsilon \|\chi\| \varepsilon \) and \( \text{meas}(\Omega_{11}) \leq 1 \).

Finally, adapting the argument for (C.7), we get
\[
\varepsilon \left| \left( (w_{12} - w_{12}^I) y, \chi_y \right)_{\Omega_{11}} \right| \leq \varepsilon \| (w_{12} - w_{12}^I) y \|_{L^\infty(\Omega_{11})} \| \chi_y \|_{L_0(\Omega_{11})} \leq C \varepsilon^{1/2} N^{-2\sigma + 1} \|\chi\| \varepsilon, \tag{C.13}
\]

Collect (C.9)–(C.13). We get
\[
\varepsilon \left| \left( (u - u^I) y, \chi_y \right)_{\Omega} \right| \leq C \varepsilon^{1/4} N^{-2} \|\chi\| \varepsilon.
\]

Combine this with (C.8) to complete the proof.

\[\square\]

**Proposition C.2.** Let \( \sigma \geq 2 \). Then
\[
\left| (c(u - u^I), \chi) \right| \leq C N^{-2} \ln^2 N \|\chi\| \varepsilon \text{ for all } \chi \in V^N.
\]

**Proof.** This follows readily from the Cauchy-Schwarz inequality and Theorem 3.1. \[\square\]

The most complicated term to bound is \( (b(u - u^I), \chi) \). Recalling the decomposition \( u = v + w_1 + w_2 + w_{12} \), we shall study the various components of \( u \) separately. Let \( \hat{w} = w_1 + w_{12} \). Integration by parts yields
\[
(b(u - u^I), \chi) = (b(v - v^I), \chi) + (b(w_2 - w_2^I), \chi)_{\Omega_{21} \cup \Omega_{22}}
- (b_x(\hat{w} - \hat{w}^I), \chi) - (b(\hat{w} - \hat{w}^I), \chi_x)
- (b_x(w_2 - w_2^I), \chi)_{\Omega_{12} \cup \Omega_{11}} - (b(w_2 - w_2^I), \chi_x)_{\Omega_{12} \cup \Omega_{11}},
\]
since \( \chi \in V^N \) vanishes on \( \Gamma \). For the terms on the right-hand side we have the following estimates which will be proved next.

\[
\left| (b_x(\tilde{w} - \tilde{w}^I), \chi) \right| + \left| (b_x(w_2 - w_2^I), \chi)_{\Omega_{12}\cup\Omega_{11}} \right| \leq C N^{-2} \ln^2 N \|\chi\|_\varepsilon \tag{C.14}
\]

\[
\left| (b(w_2 - w_2^I), \chi_x)_{\Omega_{12}\cup\Omega_{11}} \right| + \left| (b(\tilde{w} - \tilde{w}^I), \chi_x) \right| \leq C N^{-2} \ln^2 N \|\chi\|_\varepsilon, \tag{C.15}
\]

\[
\left| (b(w_2 - w_2^I), \chi)_{\Omega_{21}\cup\Omega_{22}} \right| \leq C \varepsilon^{1/4} N^{-2} \ln^2 N \|\chi\|_\varepsilon, \tag{C.17}
\]

Combining these bounds, we obtain

**Proposition C.3.** Let \( \sigma \geq 5/2 \). Then

\[
\left| (b(u - u^I), \chi) \right| \leq C N^{-2} \ln^2 N \|\chi\|_\varepsilon \quad \text{for all } \chi \in V^N.
\]

Propositions C.1–C.3 provide the bounds required in the proof of Theorem 4.3.

**Proof of (C.14).** The interpolation error bounds of Theorem 3.1 give

\[
\left| (b_x(\tilde{w} - \tilde{w}^I), \chi) \right| + \left| (b_x(w_2 - w_2^I), \chi)_{\Omega_{12}\cup\Omega_{11}} \right| \\
\leq C \left( \|w_2 - w_2^I\|_{0, \Omega_{21}\cup\Omega_{22}} + \|\tilde{w} - \tilde{w}^I\|_0 \right) \|\chi\|_0 \leq C N^{-2} \ln^2 N \|\chi\|_\varepsilon.
\]

**Proof of (C.15).**

\[
\left| (b(w_2 - w_2^I), \chi_x)_{\Omega_{12}\cup\Omega_{11}} \right| + \left| (b(\tilde{w} - \tilde{w}^I), \chi_x) \right| + \\
\leq C \left( \|w_2 - w_2^I\|_{0, \Omega_{11}} \|\chi_x\|_{0, \Omega_{11}} + \|w_2 - w_2^I\|_{0, \Omega_{12}} \|\chi_x\|_{0, \Omega_{12}} \\
+ \|\tilde{w} - \tilde{w}^I\|_{0, \Omega_{11}} \|\chi_x\|_{0, \Omega_{11}} + \|\tilde{w} - \tilde{w}^I\|_{0, \Omega_{12}} \|\chi_x\|_{0, \Omega_{12}} \\
+ \|\tilde{w} - \tilde{w}^I\|_{0, \Omega_{22}} \|\chi_x\|_{0, \Omega_{22}} + \|\tilde{w} - \tilde{w}^I\|_{0, \Omega_{21}} \|\chi_x\|_{0, \Omega_{21}} \right) \\
\leq C \left( N^{-2} \varepsilon^{1/4} N^{-2} \|\chi_x\|_{0, \Omega_{11}} + \varepsilon^{1/2} N^{-2} \ln^2 N \|\chi_x\|_{0, \Omega_{12}} \\
+ \varepsilon^{1/2} N^{-2} \ln^2 N \|\chi_x\|_{0, \Omega_{22}} + N^{-2} \ln^{1/2} N \varepsilon^{1/4} N^{-1/2} \|\chi_x\|_{0, \Omega_{21}} \right) \\
\leq C N^{-2} \ln^2 N \|\chi\|_\varepsilon,
\]

by Proposition A.1.
Proof of (C.16) and of (C.17). Let \( b_{ij} := b(x_i, y_j) \) and define a piecewise constant approximation of \( b \) by \( \bar{b} \equiv b_{ij} \) on \( \tau_{ij} \). Then

\[
(b(v - v')_x, \chi) = \sum_{\tau_{ij}} \left[ b_{ij} ((v - v')_x, \chi)_{\tau_{ij}} + ((b - \bar{b})(v - v')_x, \chi)_{\tau_{ij}} \right]
\]

\[
= \sum_{\tau_{ij}} b_{ij} H_{ij}(v, \chi) + ((b - \bar{b})(v - v')_x, \chi)
\]

\[
+ \frac{1}{12} \sum_{i=1}^{N-1} \sum_{j=1}^{N} (b_{i+1,j} h_{i+1}^2 - b_{ij} h_i^2) \int_{y_{j-1}}^{y_{j}} (\chi v_{xx})(x, y) \, dy
\]

\[=: I_1 + I_2 + I_3, \]

by Lemma 4.5.

For \( I_1 \) inequality (4.7), (2.4) and (2.2) yield

\[ |I_1| \leq C N^{-2} \left[ \|v_{xxx}\|_0 + \|v_{xyy}\|_0 \right] \|\chi\|_0 \leq C N^{-2} \|\chi\|_0, \]

while Taylor expansions give

\[ \|b - \bar{b}\|_{L_{\infty}(\tau_{ij})} \leq C (h_i + k_j) \leq C N^{-1}. \]

Thus

\[ |I_2| \leq C N^{-1} \|(v - v')_x\|_0 \|\chi\|_0 \leq C N^{-2} \|\chi\|, \]

by the Cauchy-Schwarz inequality and (A.1).

When studying \( I_3 \) for \( i \leq N/2 \), we use

\[ \int_{y_{j-1}}^{y_{j}} (\chi v_{xx})(x, y) \, dy = \sum_{k=1}^{i} \int_{\tau_{kj}} (\chi_x v_{xx} + \chi v_{xxx}) \]

which implies

\[ \sum_{i=1}^{N/2} (b_{i+1,j} h_{i+1}^2 - b_{ij} h_i^2) \int_{y_{j-1}}^{y_{j}} (\chi v_{xx})(x, y) \, dy \]

\[= \sum_{i=1}^{N/2} (b_{N/2+1,j} h_{N/2+1}^2 - b_{ij} h_i^2) \int_{\tau_{ij}} (\chi_x v_{xx} + \chi v_{xxx}). \]

For \( i > N/2 \) we employ

\[ \left| \int_{y_{j-1}}^{y_{j}} (\chi v_{xx})(x, y) \, dy \right| \leq C N \|v_{xx}\|_{L_{\infty}(\tau_{ij})} \|\chi\|_{L_1(\tau_{ij})}, \]

by (B.2) and (2.4).
Furthermore $|b_{i+1,j} - b_{ij}| \leq C h_{i+1}$ and $h_i = h_{i+1} \leq 2N^{-1}$ for $i > N/2$. Thus

$$|I_3| \leq CN^{-2} \left( \|v_{xx}\chi_x + v_{xxx}\chi\|_{L_1(\Omega_{22}\cup\Omega_{12})} + \|v_{xx}\|_{L_\infty} \|\chi\|_{L_1(\Omega_{21}\cup\Omega_{11})} \right) \tag{C.19}$$

$$\leq CN^{-2} \left( \|v_{xx}\|_{L_\infty} \left( \text{meas } \Omega_{22} \cup \Omega_{12} \right)^{1/2} \|\chi\|_{0,\Omega_{22}\cup\Omega_{12}} + \|v_{xx}\|_{L_\infty} \|\chi\|_{0,\Omega_{21}\cup\Omega_{11}} \right)$$

$$\leq CN^{-2} \ln^{1/2} N \|\chi\|_\varepsilon, \quad \text{by (2.2).}$$

Combining the estimates for the $I$’s, we get (C.17).

To prove (C.16) proceed in a similar manner:

$$\left( b(w_2 - w_2^l) \right)_x \chi_{\Omega_{21}\cup\Omega_{22}}$$

$$= \sum_{\tau_{ij} \subseteq \Omega_{21}\cup\Omega_{22}} b_{ij} H_{ij}(w_2, \chi) + \left( (b - \bar{b})(w_2 - w_2^l) \right)_x \chi_{\Omega_{21}\cup\Omega_{22}}$$

$$+ \frac{1}{12} \sum_{i=1}^{N-1} \left( \sum_{j=1}^{N/4} + \sum_{j=3N/4+1}^{N} \right) (b_{i+1,j} h_{i+1}^2 - b_{ij} h_i^2) \int_{y_{j-1}}^{y_j} (\chi w_{2,xx}) (x_i, y) dy$$

$$=: J_1 + J_2 + J_3$$

Using (4.7) and (2.2), we get

$$|J_1| \leq C \left[ N^{-2} \|w_{2,xxx}\|_0 + k^2 \|w_{2,xxg}\|_0 \right] \|\chi\|_{0,\tau_{ij}} \leq C \varepsilon^{1/4} N^{-2} \ln^2 N \|\chi\|_0.$$

For $J_2$ an argument similar to that for $J_2$ gives

$$|J_2| \leq \left| \left( (b - \bar{b})(w_2 - w_2^l) \right)_x \chi_{\Omega_{21}\cup\Omega_{22}} \right| \leq CN^{-1} \left( (w_2 - w_2^l) \right)_x \|\chi\|_{0,\Omega_{21}\cup\Omega_{22}}$$

$$\leq CN^{-2} \left( \|w_{2,xx}\|_{0,\Omega_{22}\cup\Omega_{21}} + \varepsilon^{1/2} \ln N \|w_{2,xy}\|_{0,\Omega_{22}\cup\Omega_{21}} \right) \|\chi\|_0,$$

by (3.2). Recalling (2.2), we get

$$|J_2| \leq C \varepsilon^{1/4} N^{-2} \ln N \|\chi\|_\varepsilon.$$

Finally, for $J_3$ we have, similar to (C.19)

$$|J_3| \leq CN^{-2} \left( \|w_{2,xx}\|_{L_\infty} \left( \text{meas } \Omega_{22} \right)^{1/2} \|\chi_x\|_{0,\Omega_{22}} + \|w_{2,xxx}\|_0 \|\chi\|_{0,\Omega_{22}} \right.$$

$$\left. + \|w_{2,xx}\|_{L_\infty} \left( \text{meas } \Omega_{21} \right)^{1/2} \|\chi\|_{0,\Omega_{21}} \right)$$

$$\leq C \varepsilon^{1/4} N^{-2} \ln N \|\chi\|_\varepsilon, \quad \text{by (2.2).}$$

The bounds for $J_1, J_2$ and $J_3$ give (C.17).
D  Proof of (5.5b-d)

First note that because of $0 < \beta \leq b \leq \|b\|_{L_\infty(\Omega)}$ the two semi norms $\|x\|_{L_p(\Omega)}$ and $\|b\chi_x\|_{L_p(\Omega)}$ are equivalent with constants independent of $\varepsilon$ and the mesh. This fact will be used frequently in our analysis without special reference.

Furthermore we shall use

Proposition D.1. Let $b \in W^1_\infty(\Omega)$. Then

$$\left| (b(\varphi - \varphi')_x, b\chi_x)_{\Omega_{kt}} \right| \leq C \left[ (h + k)(h\|\varphi_{xx}\|_{0,\Omega_{kt}} + k\|\varphi_{xy}\|_{0,\Omega_{kt}}) + k^2\|\varphi_{xyy}\|_{0,\Omega_{kt}} \right] \|\chi_x\|_{0,\Omega_{kt}}$$

for $k, \ell = 1, 2$.  

Proof. Let again $b_{ij} := b(x_i, y_j)$ and $\bar{b} \equiv b_{ij}$ on $\tau_{ij}$. For an arbitrary rectangle $\tau$ we have

$$(b(\varphi - \varphi')_x, b\chi_x)_\tau = \int_\tau (b^2 - \bar{b}^2)(\varphi - \varphi')_x\chi_x + \bar{b}^2 \int_\tau (\varphi - \varphi')_x\chi_x$$

The first term on the right-hand side can be bounded by a standard interpolation argument (see [2, Theorem 3.1.5]) and (3.2). For the second term use (4.6). We get

$$\left| (b(\varphi - \varphi')_x, b\chi_x)_{\tau} \right| \leq C \left[ (h + k)(h\|\varphi_{xx}\|_{0,\tau} + k\|\varphi_{xy}\|_{0,\tau}) + k^2\|\varphi_{xyy}\|_{0,\tau} \right] \|\chi_x\|_{0,\tau}$$

Summing over $\Omega_{kt}$ and applying a discrete Cauchy-Schwarz inequality, we are done. \qed

(i) Let us first consider $\Omega_{21}$ which covers the parabolic boundary layers.

$$a_{\text{stab}}(u - u', \chi)_{\Omega_{21}} = \delta_{21} \left\{ \varepsilon(\Delta u, b\chi_x)_{\Omega_{21}} + (b(u - u')_x, b\chi_x)_{\Omega_{21}} - (c(u - u'), b\chi_x)_{\Omega_{21}} \right\}.$$  

For the last term we have by Theorem 3.1 and the Cauchy-Schwarz inequality

$$\delta_{21} \left| (c(u - u'), b\chi_x)_{\Omega_{21}} \right| \leq C\delta_{21} \varepsilon^{1/4} N^{-2} \ln^{5/2} N \|b\chi_x\|_{0,\Omega_{21}} \leq C\delta_{21}^{1/2} \varepsilon^{1/4} N^{-2} \ln^{5/2} N \|\chi\|_{SD}.$$  

For the second term we proceed as follows. Let $w = w_1 + w_{12}$ and $\hat{w} = v + w_2$. Then

$$\delta_{21} \left| (b(w - w')_x, b\chi_x)_{\Omega_{21}} \right| \leq C\delta_{21} \left\{ \|w_x\|_{L_1(\Omega_{21})} \|\chi_x\|_{L_\infty(\Omega_{21})} + \|w'_x\|_{0,\Omega_{21}} \|b\chi_x\|_{0,\Omega_{21}} \right\}$$

$$\leq C\delta_{21} N \left\{ \|w_x\|_{L_1(\Omega_{21})} \varepsilon^{-1/4} \ln^{-1/2} N \|\chi_x\|_{0,\Omega_{21}} + \varepsilon^{1/4} \ln^{1/2} N \|w'_j\|_{L_\infty(\Omega_{21})} \|b\chi_x\|_{0,\Omega_{21}} \right\}$$

$$\leq C\delta_{21}^{1/2} \varepsilon^{1/4} N^{-\sigma+1} \ln^{1/2} N \|\chi\|_{SD},$$

where $\sigma$ is such that $\sigma > \frac{\varepsilon^{1/4} \ln^{1/2} N}{\varepsilon^{1/4} N^{\sigma-1}}$.  

\[\text{Proof.}\]
by (B.1), (B.3) and (2.1). For the term involving \( \tilde{w} \) use Proposition D.1.

\[
\delta_{21} \left| \left( b(\tilde{w} - \tilde{w}^t)_x, b\chi_x \right)_{\Omega_{21}} \right| \leq C\delta_{21} N^{-2} \left[ \|\tilde{w}_{xx}\|_{0, \Omega_{21}} + \varepsilon^{1/2} \ln N \|\tilde{w}_{xy}\|_{0, \Omega_{21}} \right.
\]
\[
\left. + \varepsilon \ln^2 N \|\tilde{w}_{xyy}\|_{0, \Omega_{21}} \right] \|\chi_x\|_{0, \Omega_{21}}
\]
\[
\leq C\delta_{21}^{1/2} \varepsilon^{1/4} N^{-2} \ln^2 N \|\chi\|_S D,
\]

When bounding \( (\Delta u, b\chi_x) \) we use the splitting \( u = w + \tilde{w} \) again.

\[
\delta_{21} \varepsilon \left| (\Delta w, b\chi_x)_{\Omega_{21}} \right| \leq \delta_{21} \varepsilon \|\Delta w\|_{L_1(\Omega_{21})} \|b\chi_x\|_{L_\infty(\Omega_{21})}
\]
\[
\leq C\delta_{21} \varepsilon N^{-\sigma} \varepsilon^{-1/2} \ln N (N\varepsilon^{-1/4} \ln^{-1/2}) \|b\chi_x\|_{0, \Omega_{21}}
\]
\[
\leq C\delta_{21}^{1/2} \varepsilon^{1/4} N^{-\sigma+1} \ln^{1/2} N \|\chi\|_S D,
\]

by (B.3) and (2.1).

Next

\[
(\Delta \tilde{w}, b\chi_x)_{\Omega_{21}} + (\Delta \tilde{w}, b\chi_x)_{\Omega_{22}} = - (b\Delta \tilde{w})_x, \chi)_{\Omega_{21} \cup \Omega_{22}}
\]

and (2.1) imply

\[
\delta_{21} \varepsilon \left| (\Delta \tilde{w}, b\chi_x)_{\Omega_{21}} \right| \leq C\delta_{21} \varepsilon \left( \varepsilon^{-3/4} \|\chi\|_{0, \Omega_{21} \cup \Omega_{22}} + \varepsilon^{-1/4} \ln^{1/2} N \|\chi_x\|_{0, \Omega_{22}} \right)
\]
\[
\leq C\delta_{21}^{1/2} \varepsilon^{1/4} \ln^{1/2} N \|\chi\|_\varepsilon.
\]

Collection the above results, we get (5.5b).

\( (ii) \) The proof of (5.5c), i.e. for the region of the exponential layer, uses precisely the same argument. The only difference is in the bounds for the various (semi-)norms of \( u \) and its components.

\( (iii) \) Finally consider \( \Omega_{22} \). By Theorem 3.1 we have

\[
\left| (c(u - u^t), b\chi_x)_{\Omega_{22}} \right| \leq C\varepsilon^{3/4} N^{-2} \ln^3 N \|\chi_x\|_0.
\]

Proposition D.1 and a direct calculation of the norms of \( u \) involved give

\[
\left| (b(u - u^t)_x, b\chi_x)_{\Omega_{22}} \right| \leq C\varepsilon^{-1/4} N^{-2} \ln^2 N \|\chi_x\|_0.
\]

Finally

\[
\varepsilon \left| (\Delta u, b\chi_x)_{\Omega_{22}} \right| \leq C\varepsilon^{-1/4} \ln N \|\chi_x\|_0.
\]

This is the dominant term on \( \Omega_{22} \).

Remark that \( \|\chi_x\|_0 \leq \varepsilon^{-1/2} \|\chi\|_SD \). We get (5.5d).