A finite element analysis of a coupled system of singularly perturbed reaction-diffusion equations

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Abstract

We consider a system of two coupled reaction-diffusion equations. When the parameters multiplying the second-order derivatives in the equations are small, their solutions exhibit boundary layers. Moreover, when the parameters are of different magnitudes, two distinct but overlapping boundary layers are present. We study a finite element discretization on general layer-adapted meshes including the frequently studied Shishkin mesh and the Bakhvalov mesh. Supporting numerical results are presented.

Keywords: Reaction diffusion, singular perturbation, solution decomposition, Shishkin mesh.

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1. Introduction

We consider a finite element discretization of two coupled singularly perturbed reaction-diffusion problems

$$\begin{aligned} -\varepsilon^2 u_1''(x) + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) &= f_1(x), \\ -\mu^2 u_2''(x) + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) &= f_2(x) \end{aligned} \tag{1a}$$

where $x \in (0,1)$ with the boundary conditions

$$u_1(0) = u_2(0) = u_1(1) = u_2(1) = 0. \tag{1b}$$

The parameters ε and μ are small, positive constants in the range $(0,1]$. Without loss of generality we shall assume that

$$0 < \varepsilon \leq \mu \ll 1. \tag{2}$$

The solution $\boldsymbol{u} = (u_1, u_2)^T$ to (1) has overlapping boundary layers of width $\mathcal{O}(\varepsilon \ln \varepsilon)$ and $\mathcal{O}(\mu \ln \mu)$ at $x = 0$ and $x = 1$. We shall assume that $A = \{a_{ij}\}_{i,j=1}^{2}$ is an L_0 matrix with

$$\min_{[0,1]} \{a_{11} + a_{12}, a_{21} + a_{22}\} > \alpha^2, \tag{3}$$

i. e., A is an M-matrix whose inverse is bounded by α^{-2} in the maximum norm.

Shishkin [11] considers a similar problem: a system of two parabolic equations on a strip. He cites the modelling of mass transfer processes in multicomponent systems as a motivation. The study is motivated by models for certain complicated chemical reactions. He classifies the problems according to

(i) $0 < \varepsilon = \mu \ll 1$, (ii) $0 < \varepsilon \ll \mu = 1$, (iii) $0 < \varepsilon \leq \mu \ll 1$,

and presents a piecewise-uniform (Shishkin) mesh for each case. He shows that a standard finite difference method on these meshes converge uniformly with respect to the parameters in the maximum norm. The orders of convergence are $\mathcal{O}\left(N^{-1} \ln N\right)$ for (i), $\mathcal{O}\left(N^{-2/5}\right)$ for (ii) and $\mathcal{O}\left(N^{-1/4}\right)$ for the most general case (iii).

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Matthews et al. [6, 7] consider cases (i) and (ii) for the problem as given in (1). Using the classical finite difference technique, they obtain almost first-order parameter uniform convergence (in the maximum norm) on a Shishkin mesh. For case (ii), the same authors [8] improve this result to show almost second-order convergence.

Madden and Stynes [5] also consider a finite difference approach for (1), paying particular attention to case (iii). Their motivation for studying the systems is based on a two-equation turbulence model. Following [11], they study a finite difference method on a piecewise uniform mesh with a transition point for each of the layers. The method is shown to be almost first-order accurate in the maximum norm.

This present study is devoted to a finite element method for (1) on general meshes not only the Shishkin mesh. We derive a general criterion that guarantees parameter uniform convergence in the energy norm naturally associated with the weak formulation of (1) and in the L_2 norm.

An outline of the paper is as follows. In Section 2 a decomposition of the exact solution is presented and studied. We describe an appropriate finite element discretization in Section 3 and give interpolation and approximation error bounds for arbitrary meshes. In Section 4 these results are applied to establish the parameter uniform convergence of the FEM on Bakhvalov and on Shishkin meshes. The paper concludes with supporting numerical results.

Notation: Throughout the paper C will denote a generic positive constant that is independent of the perturbation parameters ε and μ and of the number N of mesh nodes.

2. Properties of the exact solution

For the construction of layer-adapted meshes and the analysis of numerical methods it is necessary to have precise knowledge of the behaviour of the exact solution to be approximated and its derivatives.

Lemma 1. Let $\boldsymbol{u}$ be the solution to (1). Then there exists a constant C, such that for all $x \in [0,1]$, we have

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}_0 + \boldsymbol{w}_1$$

where the regular solution component $\boldsymbol{v}$ satisfies

$$\left|v_i''(x)\right| \le C \quad \text{for} \quad i = 1, 2, \tag{4a}$$

while the layer components $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ satisfy

$$|w''_{0,1}(x)| \le C\left(\varepsilon^{-2}e^{-\alpha x/\varepsilon} + \mu^{-2}e^{-\alpha x/\mu}\right), \qquad |w''_{0,2}(x)| \le C\mu^{-2}e^{-\alpha x/\mu}, \tag{4b}$$

$$|w''_{1,1}(x)| \le C\left(\varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon} + \mu^{-2}e^{-\alpha(1-x)/\mu}\right), \qquad |w''_{1,2}(x)| \le C\mu^{-2}e^{-\alpha(1-x)/\mu}. \tag{4c}$$

Proof. Lemma 4 of [5] gives that

$$|u''_1(x)| \le C\left\{1 + \varepsilon^{-2}\left(e^{-\alpha x/\varepsilon} + e^{-\alpha(1-x)/\varepsilon}\right) + \mu^{-2}\left(e^{-\alpha x/\mu} + e^{-\alpha(1-x)/\mu}\right)\right\}$$

and

$$|u''_2(x)| \le C\left\{1 + \mu^{-2}\left(e^{-\alpha x/\mu} + e^{-\alpha(1-x)/\mu}\right)\right\}.$$

We first derive the splitting for u_1, considering separately the cases $\mu \le 1/e$ and $\mu > 1/e$.

(i) Let $\mu \le 1/e$. Then following the technique in [3], we set $x^* := 2\mu\alpha^{-1}\ln(1/\mu)$ and

$$u_1 := v_1 \quad \text{on} \quad (x^*, 1-x^*).$$

Since $e^{-\alpha x^*/\mu} = \mu^2$ and $e^{-\alpha x^*/\varepsilon} \le \varepsilon^2$ we get $|v''_1(x)| \le C$ on $(x^*, 1-x^*)$ and v_1 extends to a smooth function on $(0,1)$ that satisfies (4a). Now consider

$$w_{0,1}(x) := \begin{cases} u_1(x) - v_1(x) & \text{on} \quad [0, x^*], \\ 0 & \text{on} \quad [x^*, 1]. \end{cases}$$

This function satisfies (4b). Finally we set $w_{1,1} := u_1 - v_1 - w_{0,1}$. It satisfies (4c).

(ii) If $\mu > 1/e$ then there exists a constant C such that

$$|u''_1(x)| \le C\left\{1 + \varepsilon^{-2}\left(e^{-\alpha x/\varepsilon} + e^{-\alpha(1-x)/\varepsilon}\right)\right\}$$

and we repeat the above construction with $x^* := 2\varepsilon\alpha^{-1}\ln(1/\mu)$.

The constructions of u_2, $w_{0,2}$ and $w_{1,2}$ are similar. ■

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3. Analysis of the finite element method

As usual with finite element discretization we consider the weak formulation: Find $\boldsymbol{u} \in H_0^1(0,1)^2$ with

$$B(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{f}(\boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{v} \in H_0^1(0,1)^2,$$

with

$$B(\boldsymbol{u},\boldsymbol{v}) := \varepsilon^2(u_1', v_1') + \mu^2(u_2', v_2') + (a_{11}u_1 + a_{12}u_2, v_1) + (a_{21}u_1 + a_{22}u_2, v_2)$$

and

$$\boldsymbol{f}(\boldsymbol{v}) := (f_1, v_1) + (f_2, v_2),$$

where $(v,w) = \int_0^1 vw$ denotes the standard scalar product in $L_2(0,1)$.

A natural norm on $H_0^1(0,1)^2$ associated with the bilinear form $B(\cdot,\cdot)$ is the energy norm

$$|||\boldsymbol{v}|||^2 = \varepsilon^2|v_1|_1^2 + \mu^2|v_2|_1^2 + \alpha^2\left(\|v_1\|_0^2 + \|v_2\|_0^2\right),$$

where by $\|v\|_0 := (v,v)^{1/2}$ we denote the standard norm on $L_2(0,1)$, while $|v|_1 := \|v'\|_0$ is the usual semi-norm on $H_0^1(0,1)$. We also use the notation $\|\boldsymbol{v}\|_0 = \left\{\|v_1\|_0^2 + \|v_2\|_0^2\right\}^{1/2}$ for the norm in $L_2(0,1)^2$. Our assumption (3) on A implies that for arbitrary $x \in (0,1)$

$$\boldsymbol{\xi}^T A \boldsymbol{\xi} \geq \alpha^2 \boldsymbol{\xi}^T \boldsymbol{\xi} \quad \text{for all} \quad \boldsymbol{\xi} \in I\!\!R^2.$$

It follows the bilinear form B is coercive with respect to $|||\cdot|||$, i. e.,

$$|||\boldsymbol{v}|||^2 \leq B(\boldsymbol{v},\boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{v} \in H_0^1(0,1)^2. \tag{5}$$

3.1. Discretization

Let $\Omega : 0 = x_0 < x_1 < \ldots < x_N = 1$ be an arbitrary mesh with local step sizes $h_i = x_i - x_{i-1}$. Let $V \subset H_0^1(0,1)$ be the space of piecewise linear functions on Ω that vanish at $x = 0$ and $x = 1$. Then our discretization is: Find $\boldsymbol{U} \in V^2$ such that

$$B(\boldsymbol{U},\boldsymbol{v}) = \boldsymbol{f}(\boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{v} \in V^2. \tag{6}$$

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For simplicity in our analysis we shall assume that all integrals can be evaluated exactly. If this is not the case appropriate quadrature formulae have to be used.

The convergence analysis for the scheme starts at the triangle inequality

$$|||\boldsymbol{u}-\boldsymbol{U}||| \le |||\boldsymbol{u}-\boldsymbol{u}^I||| + |||\boldsymbol{u}^I-\boldsymbol{U}|||, \tag{7}$$

where $\boldsymbol{u}^I$ denotes the piecewise linear interpolant to $\boldsymbol{u}$ on Ω. Because of the Galerkin orthogonality relation between $\boldsymbol{u}$ and $\boldsymbol{U}$, $B(\boldsymbol{U}-\boldsymbol{u},\boldsymbol{u}^I-\boldsymbol{U})=0$. Then from the coercivity (5) of $B(\cdot,\cdot)$, we have that

$$\begin{aligned} |||\boldsymbol{u}^I-\boldsymbol{U}|||^2 &\le B(\boldsymbol{u}^I-\boldsymbol{U},\boldsymbol{u}^I-\boldsymbol{U}) \\ &= \big(a_{11}(u_1^I-u_1)+a_{12}(u_2^I-u_2),u_1^I-U_1\big) \\ &\qquad + \big(a_{21}(u_1^I-u_1)+a_{22}(u_2^I-u_2),u_2^I-U_2\big)\,, \end{aligned}$$

where we have used integration by parts. Thus

$$|||\boldsymbol{u}^I-\boldsymbol{U}|||^2 \le C\|\boldsymbol{u}^I-\boldsymbol{u}\|_0\|\boldsymbol{u}^I-\boldsymbol{U}\|_0 \le C\|\boldsymbol{u}^I-\boldsymbol{u}\|_0|||\boldsymbol{u}^I-\boldsymbol{U}|||.$$

We get

$$|||\boldsymbol{u}^I-\boldsymbol{U}||| \le C\|\boldsymbol{u}^I-\boldsymbol{u}\|_0. \tag{8}$$

Looking at (7) and (8), we see that we need bounds only for the interpolation error. These are derived in the next section.

3.2. Interpolation error bounds

Let $\varphi \in C^2(0,1)$ be arbitrary. Then using a Taylor expansion at x_i, we can write the interpolation error for $x \in [x_{i-1},x_i]$ as

$$(\varphi^I-\varphi)(x) = \frac{x_i-x}{h_i}\int_{x_i}^{x_{i-1}} \varphi''(\sigma)(x_{i-1}-\sigma)\,d\sigma - \int_{x_i}^{x} \varphi''(\sigma)(x-\sigma)\,d\sigma.$$

Thus

$$\big|(\varphi^I-\varphi)(x)\big| \le 2\int_{x_{i-1}}^{x_i} \big|\varphi''(\sigma)\big|(\sigma-x_{i-1})\,d\sigma. \tag{9}$$

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Now we study the interpolation error for the solution of (1). Let

$$E_\nu(x) := \nu^{-1}\left(e^{-\alpha x/2\nu} + e^{-\alpha(1-x)/2\nu}\right).$$

Lemma 2. Suppose $\boldsymbol{u}$ can be decomposed as in Lemma 1. Then the interpolation error $\boldsymbol{u} - \boldsymbol{u}^I$ satisfies

$$\max_{[x_{i-1},x_i]}\left|(u_1 - u_1^I)(x)\right| \le C\Big\{\int_{x_{i-1}}^{x_i} (1 + E_\varepsilon(x) + E_\mu(x))\,dx\Big\}^2,$$

$$\max_{[x_{i-1},x_i]}\left|(u_2 - u_2^I)(x)\right| \le C\Big\{\int_{x_{i-1}}^{x_i} (1 + E_\mu(x))\,dx\Big\}^2,$$

$$\left|u_1 - u_1^I\right|_1 \le C\varepsilon^{-1/2} \max_{i=1,\dots,N} \int_{x_{i-1}}^{x_i} (1 + E_\varepsilon(x) + E_\mu(x))\,dx$$

and

$$\left|u_2 - u_2^I\right|_1 \le C\mu^{-1/2} \max_{i=1,\dots,N} \int_{x_{i-1}}^{x_i} (1 + E_\mu(x))\,dx.$$

Proof. For the sake of simplicity we study $u_2 - u_2^I$ only. Let $x \in [x_{i-1}, x_i]$ be arbitrary. A triangle inequality gives

$$\left|(u_2 - u_2^I)(x)\right| \le \left|(v_2 - v_2^I)(x)\right| + \left|(w_{0,2} - w_{0,2}^I)(x)\right| + \left|(w_{1,2} - w_{1,2}^I)(x)\right|. \tag{10}$$

The bounds for $\boldsymbol{v}''$ of Lemma 1 and (9) give

$$\left|(v_2 - v_2^I)(x)\right| \le Ch_i^2 = C\left\{\int_{x_{i-1}}^{x_i} 1\,dx\right\}^2, \tag{11}$$

while for $w_{0,2} - w_{0,2}^I$ we have

$$\left|(w_{0,2} - w_{0,2}^I)(x)\right| \le C\int_{x_{i-1}}^{x_i} \mu^{-2} e^{-\alpha\sigma/\mu}(\sigma - x_{i-1})\,d\sigma.$$

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To bound the right-hand side we use the inequality

$$\int_{x_{i-1}}^{x_i} g(\sigma)(\sigma - x_{i-1})\, d\sigma \leq \frac{1}{2}\left\{\int_{x_{i-1}}^{x_i} g(\sigma)^{1/2}\, d\sigma\right\}^2,$$

which holds true for any positive monotonically decreasing function g on $[x_{i-1}, x_i]$; see [2]. This can be easily verified by considering the two integrals as functions of the upper integration limit. We get

$$\left|(w_{0,2} - w_{0,2}^I)(x)\right| \leq C\left\{\int_{x_{i-1}}^{x_i} \mu^{-1} e^{-\alpha\sigma/2\mu}\, d\sigma\right\}^2. \tag{12}$$

Because of symmetry we have

$$\left|(w_{1,2} - w_{1,2}^I)(x)\right| \leq C\left\{\int_{x_{i-1}}^{x_i} \mu^{-1} e^{-\alpha(1-\sigma)/2\mu}\, d\sigma\right\}^2. \tag{13}$$

Combining (10)–(13) we get

$$\left|(u_2 - u_2^I)(x)\right| \leq C\left\{\int_{x_{i-1}}^{x_i} (1 + E_\mu(\sigma))\, d\sigma\right\}^2.$$

The maximum-norm estimate of the Lemma for $u_2 - u_2^I$ follows.

For the error in the H^1-semi norm, integration by parts yields

$$\left|u_2^I - u_2\right|_1^2 = \int_0^1 \left((u_2^I - u_2)'\right)^2(x)\, dx = -\int_0^1 (u_2^I - u_2)(x) u_2''(x)\, dx.$$

Using the derivative bounds of Lemma 1, we get

$$\left|u_2^I - u_2\right|_1 \leq C\mu^{-1/2} \max_{x\in[0,1]} \left|(u_2^I - u_2)(x)\right|^{1/2}.$$

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Remark 1. Let

$$\vartheta(\Omega) = \max_{i=1,\dots,N} \int_{x_{i-1}}^{x_i} \left(1 + E_\varepsilon(x) + E_\mu(x)\right) dx.$$

Then the results of Lemma 2 yield the following interpolation error bounds:

$$\|\boldsymbol{u} - \boldsymbol{u}^I\|_0 \le C\vartheta(\Omega)^2 \quad \text{and} \quad |||\boldsymbol{u} - \boldsymbol{u}^I||| \le C\big(\mu^{1/2} + \vartheta(\Omega)\big)\vartheta(\Omega).$$

3.3. Discretization error

We can now state our results for the discretization error of the finite element method when applied to (1).

Theorem 1. Suppose $\boldsymbol{u}$ can be decomposed as in Lemma 1. Then the discretization error satisfies the uniform estimates

$$\left\|\boldsymbol{u} - \boldsymbol{U}\right\|_0 \le C\vartheta(\Omega)^2, \quad |||\boldsymbol{u}^I - \boldsymbol{U}||| \le C\vartheta(\Omega)^2 \quad \text{and} \quad |||\boldsymbol{u} - \boldsymbol{U}||| \le C\vartheta(\Omega).$$

Proof. These results follow readily from (8) and from Remark 1. ■

4. Layer-adapted meshes

We now employ the result of Theorem 1 to analyse the FEM discretization (6) on two standard layer-adapted meshes.

4.1. Bakhvalov meshes

Bakhvalov meshes [1] for the discretization of (1) may be regarded as generated by equidistributing the function

$$\begin{aligned} M_{Ba}(x) = \max \Bigg\{ & \frac{K_\varepsilon}{\varepsilon} \exp\left(-\frac{\alpha x}{\varepsilon\sigma_\varepsilon}\right), \frac{K_\mu}{\mu} \exp\left(-\frac{x\alpha}{\mu\sigma_\mu}\right), \\ & 1, \frac{K_\varepsilon}{\varepsilon} \exp\left(-\frac{\alpha(1-x)}{\varepsilon\sigma_\varepsilon}\right), \frac{K_\mu}{\mu} \exp\left(-\frac{\alpha(1-x)}{\mu\sigma_\mu}\right) \Bigg\}, \end{aligned}$$

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with positive constants $K_\varepsilon, K_\mu, \sigma_\varepsilon$ and σ_μ, i. e., the mesh points x_i are chosen such that

$$\int_{x_{i-1}}^{x_i} M_{Ba}(x)dx = \frac{1}{N}\int_0^1 M_{Ba}(x)dx.$$

The parameters K_ε and K_μ determine the number of mesh points used to resolve the respective layer, while σ_ε and σ_μ determine the grading of the mesh inside them.

Clearly we have $1+E_\varepsilon(x)+E_\mu(x) \le CM_{Ba}(x)$ if $\sigma_\varepsilon, \sigma_\mu \ge 2$ and $\int_0^1 M_{Ba}(x)dx \le C$. Using Theorem 1, we can conclude

$$\left\|\boldsymbol{u}-\boldsymbol{U}\right\|_0 \le CN^{-2}, \quad |||\boldsymbol{u}^I-\boldsymbol{U}||| \le CN^{-2} \text{ and } |||\boldsymbol{u}-\boldsymbol{U}||| \le C\left(\mu^{1/2}+N^{-1}\right)N^{-1} \tag{14a}$$

or uniformly, in ε and μ,

$$|||\boldsymbol{u}-\boldsymbol{U}||| \le CN^{-1}. \tag{14b}$$

4.2. Shishkin meshes

Shishkin meshes [9, 10] are frequently studied because of their simplicity—they are piecewise uniform. We describe a possible construction for problem (1). Let $q_\varepsilon, q_\mu \in (0,1)$ with $2q_\varepsilon + 2q_\mu < 1$ and $\sigma_\varepsilon, \sigma_\mu > 0$ be mesh parameters. We set

$$\lambda_\mu = \min\left\{q_\varepsilon + q_\mu, \frac{\sigma_\mu \mu}{\alpha}\ln N\right\} \quad \text{and} \quad \lambda_\varepsilon = \min\left\{\frac{q_\varepsilon}{q_\varepsilon+q_\mu}\lambda_\mu, \frac{\sigma_\varepsilon \varepsilon}{\alpha}\ln N\right\}.$$

Assuming that $q_\varepsilon N$ and $q_\mu N$ are integers, we divide each of the intervals $[0,\lambda_\varepsilon]$ and $[1-\lambda_\varepsilon,1]$ into $q_\varepsilon N$ subintervals, while $[\lambda_\varepsilon,\lambda_\mu]$ and $[1-\lambda_\mu,1-\lambda_\varepsilon]$ are divided into $q_\mu N$ and $[\lambda_\mu,1-\lambda_\mu]$ into $(1-2q_\varepsilon-2q_\mu)N$ subintervals.

For the mesh constructed this way we can adapt the technique from [3, 4] to get

$$\vartheta(\Omega) \le C\left\{N^{-\sigma_\varepsilon} + N^{-\sigma_\mu} + N^{-1}\ln N\right\}.$$

Therefore Theorem 1 yields the uniform error bounds: choose the mesh parameters so that

$$\sigma_\varepsilon \ge 2, \quad \sigma_\mu \ge 2, \tag{15}$$

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then

$$\left\|\boldsymbol{u}-\boldsymbol{U}\right\|_0 \le C\left(N^{-1}\ln N\right)^2, \quad |||\boldsymbol{u}^I-\boldsymbol{U}||| \le C\left(N^{-1}\ln N\right)^2, \tag{16a}$$

and

$$|||\boldsymbol{u}-\boldsymbol{U}||| \le CN^{-1}\ln N. \tag{16b}$$

The mesh use by Madden and Stynes [5] can be obtained by taking $q_\varepsilon = q_\mu = 1/8$, and $\sigma_\varepsilon = \sigma_\mu = 1$. The original mesh of Shishkin [11] is constructed slightly differently: λ_μ depends on $\varepsilon + \mu$.

5. Numerical results

In this section we verify experimentally our convergence results be considering the numerical solution of a constant and a variable coefficient problem.

Usually, the exact solutions of the problems are not available, so we estimate the accuracy of the numerical solution by comparing it to the numerical solution computed on a finer mesh.

Indicating by $U^N_{\varepsilon,\mu}$ that the numerical approximation depends on N, ε and μ, we estimate the uniform error by

$$\eta^N := \max_{\varepsilon,\mu=1,10^{-1},\dots,10^{-12}} \left\|U^N_{\varepsilon,\mu} - \tilde{U}^{8N}_{\varepsilon,\mu}\right\|,$$

where $\tilde{U}^{8N}_\varepsilon$ is the approximate solution of the FEM on a mesh obtained by bisecting the original mesh three times, i. e., a mesh that is 8 times finer. The rates of convergence are computed using the standard formula $r^N = \log_2\left(\eta^N/\eta^{2N}\right)$.

Example 1

$$\begin{aligned} -\varepsilon u_1'' + 2u_1 - u_2 &= 1, \quad u_1(0) = u_1(1) = 0, \\ -\mu u_2'' - u_1 + 2u_2 &= 1, \quad u_2(0) = u_2(1) = 0. \end{aligned}$$

This problem satisfies (3) for any $\alpha \in (0,1)$.

Table 1 contains the results of our test computations for the Bakhvalov mesh of Section 4.1, and illustrates clearly the conclusions (14) of that section: we obtain first-order uniform convergence in the continuous energy norm and second-order uniform convergence in the discrete energy norm. Observe also that second-order convergence is achieved in the maximum norm.

	$\|u - U\|_\infty$		$\|\|\|u - U\|\|\|$		$\|\|\|u^I - U\|\|\|$	
N	error	rate	error	rate	error	rate
48	8.309e-3	1.95	3.427e-2	1.00	2.710e-3	2.01
96	2.154e-3	1.97	1.718e-2	0.97	6.737e-4	2.00
192	5.506e-4	1.96	8.762e-3	1.00	1.687e-4	2.00
384	1.419e-4	1.97	4.381e-3	1.00	4.220e-5	2.00
768	3.633e-5	2.00	2.196e-3	1.00	1.055e-5	2.00
1536	9.108e-6	2.00	1.100e-3	1.00	2.638e-6	2.00
3072	2.281e-6	2.00	5.506e-4	1.00	6.594e-7	2.00
6144	5.709e-7	—	2.753e-4	—	1.649e-7	—

Table 1: Example 1, Bakhvalov mesh

The results of our computations on the Shishkin mesh of Section 4.2 are shown in Table 2. Again, the conclusions (16) of that section are verified: there is (almost) first-order uniform convergence in the continuous energy norm and of the (almost) second-order uniform convergence in the discrete energy norm. We note that there is (almost) second-order convergence in the maximum norm.

NB. In our test computations we have in no way tried to optimize the parameters defining the mesh—we have merely ensured that the critical parameters σ_ε and σ_μ are chosen to satisfy (15). We have taken $\sigma_\varepsilon = \sigma_\mu = 2$.

Example 2 The following problem is adapted from Matthews et al. [8].

$$\begin{aligned} -\varepsilon u_1'' + 2(x+1)^2 u_1 - (1+x^3) u_2 &= 2\exp(x), \quad u_1(0) = u_1(1) = 0, \\ -\mu u_2'' - 2\cos(\pi x/4) u_1 + (1+\sqrt{2})\exp(1-x) u_2 &= 10x + 1, \quad u_2(0) = u_2(1) = 0. \end{aligned}$$

Results are presented in Table 3 for the Bakhvalov mesh, and in Table 4 for the Shishkin mesh. Again the theoretical results of Section 4 are confirmed.

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	$\|u-U\|_\infty$		$|||u-U|||$		$|||u^I-U|||$	
N	error	rate	error	rate	error	rate
48	1.091e-1	1.23	8.756e-2	1.21	1.822e-2	1.78
96	4.666e-2	1.42	3.773e-2	1.10	5.299e-3	1.97
192	1.744e-2	1.55	1.754e-2	1.03	1.353e-3	1.69
384	5.965e-3	1.63	8.587e-3	1.01	4.183e-4	1.67
768	1.927e-3	1.69	4.269e-3	0.94	1.310e-4	1.70
1536	5.992e-4	1.72	2.225e-3	0.87	4.026e-5	1.73
3072	1.814e-4	1.75	1.216e-3	0.88	1.214e-5	1.75
6144	5.381e-5	—	6.600e-4	—	3.601e-6	—

Table 2: Example 1, Shishkin mesh

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	$\|u-U\|_\infty$		$\|\|\|u-U\|\|\|$		$\|\|\|u^I-U\|\|\|$	
N	error	rate	error	rate	error	rate
48	6.921e-2	1.88	1.657e-1	0.98	1.477e-2	2.00
96	1.886e-2	1.93	8.377e-2	0.96	3.700e-3	2.01
192	4.950e-3	1.92	4.311e-2	1.00	9.210e-4	2.00
384	1.304e-3	1.95	2.157e-2	0.99	2.302e-4	2.00
768	3.385e-4	1.97	1.082e-2	1.00	5.765e-5	2.00
1536	8.618e-5	1.99	5.423e-3	1.00	1.442e-5	2.00
3072	2.170e-5	1.99	2.717e-3	1.00	3.607e-6	2.00
6144	5.445e-6	—	1.358e-3	—	9.019e-7	—

Table 3: Example 2, Bakhvalov mesh

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	$\|u-U\|_\infty$		$|||u-U|||$		$|||u^I-U|||$	
N	error	rate	error	rate	error	rate
48	7.985e-1	1.17	4.342e-1	1.15	1.164e-1	1.49
96	3.540e-1	1.34	1.961e-1	1.08	4.147e-2	1.81
192	1.398e-1	1.42	9.287e-2	1.02	1.180e-2	1.64
384	5.222e-2	1.56	4.569e-2	1.01	3.799e-3	1.66
768	1.767e-2	1.65	2.275e-2	0.94	1.202e-3	1.72
1536	5.629e-3	1.71	1.182e-2	0.87	3.658e-4	1.75
3072	1.725e-3	1.74	6.460e-3	0.88	1.086e-4	1.78
6144	5.152e-4	—	3.507e-3	—	3.162e-5	—

Table 4: Example 2, Shishkin mesh

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