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Herausgeber: Der Rektor

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MATH-NM-06-2007
September 2007

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September 1, 2007

Abstract

Some recently developed finite element stabilizations of convection-diffusion problems generate in 1D five-point difference schemes. Because there are only few results on four- and five-point schemes in the literature (in contrast to three-point schemes) we discuss some properties of such schemes with special emphasis on the choice of free parameters for a singularly perturbed problem to avoid oscillations.

AMS subject classification (2000): 65L10, 65L12, 65L60.

1 Introduction

Let us consider the singularly perturbed boundary value problem

$$-\varepsilon u'' - bu' = f \quad \text{on} \quad (0,1), \;\; u(0) = u(1) = 0 \tag{1.1}$$

under the assumptions $0 < \varepsilon \ll 1$, $b(x) \geq \beta > 0$ and its discretization with finite differences or low order, in general linear, finite elements. It is well known that the use of the central difference scheme or the Galerkin method with linear elements leads to wild oscillations on standard meshes due to the existence of an exponential boundary layer at $x = 0$ unless the mesh width is as small as ε which makes no sense from the practical point of view.

Therefore, it is quite standard to apply some kind of upwinding in the finite difference framework or to stabilize the Galerkin finite element method (here we do not discuss other approaches, for instance, upwinding in the finite volume method).

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For simplicity, we restrict ourself to uniform meshes with the mesh width h and the mesh points $x_i := ih$, $i = 0, 1, \cdots, N$ with $x_N = 1$, assuming always $\varepsilon \leq C\,h$ and moderate C. We use the difference operators

$$D^+u_i := \frac{u_{i+1} - u_i}{h}, \quad D^-u_i := \frac{u_i - u_{i-1}}{h}, \quad D_0u_i := \frac{u_{i+1} - u_{i-1}}{2h}.$$

Then, simple upwinding reads

$$\begin{aligned} -\varepsilon D^+D^-u_i - b_iD^+u_i &= f_i, \quad i = 1, ..., N-1, \\ u_0 = u_N = 0 & \end{aligned} \tag{1.2}$$

the corresponding simplest stabilization of the Galerkin method based on linear elements

$$\varepsilon({u_h}', {v_h}') - (b{u_h}', v_h) + \frac{h}{2}(b{u_h}', {v_h}') = (f, v_h). \tag{1.3}$$

Remark that the stabilization term $\frac{h}{2}(b{u_h}', {v_h}')$ in the finite element framework has its analogous in the finite difference language, one can rewrite (1.2) in the form

$$-\varepsilon\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - b_i\frac{u_{i+1} - u_{i-1}}{2h} + b_i\frac{h}{2}\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i. \tag{1.4}$$

Because (1.2) and (1.3) are first order methods, one is interested to construct higher order methods. Here we do not discuss the midpoint upwind finite difference scheme [9] or streamline diffusion based on linear elements [10] because we want to compare approaches leading to four- or five-point schemes. For simplicity, from now we assume b to be constant in the given problem (1.1).

Our renewed interest in such schemes comes from the fact that several recently developed finite element stabilizations of convection-diffusion problems [3],[4],[6] generate in 1D such five-point schemes. Moreover, in most cases the optimal tuning of parameters involved in the stabilization term is an open problem.

2 Finite difference schemes

A well-known four-point scheme for solving (1.1) is given by

$$\left.\begin{aligned} -\varepsilon D^+D^-u_i - bD_0u_i & \\ +\frac{b\lambda}{h}(-u_{i-1} + 3u_i - 3u_{i+1} + u_{i+2}) &= f_i, \qquad i = 1, ..., N-2, \\ -\varepsilon D^+D^-u_{N-1} - bD^+u_{N-1} &= f_{N-1}, \\ u_0 = u_N &= 0 \end{aligned}\right\} \tag{2.1}$$

$\lambda > 0$ is a parameter. Remark that the stabilization term is an $0(h^2)$ perturbation of the third order derivative:

$$\frac{-u(x_{i-1}) + 3u(x_i) - 3u(x_{i+1}) + u(x_{i+2})}{h^3} = u'''(x_i) + 0(h).$$

It is quite interesting that the scheme (2.1) is for certain values of the parameter λ inverse-monotone.
To see that let us introduce matrices M_1, M_2 of format $(N+1)\times(N+1)$ by

$$M_1(\lambda) = M_1 = \begin{bmatrix} 1 & 0 & 0 & & & 0 \\ \alpha & \alpha+\beta & -\delta & & & \\ & \ddots & \ddots & \ddots & & \\ & & \alpha & \alpha+\beta & -\delta & 0 \\ & & 0 & r & s & 0 \\ 0 & & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & -1 & & 0 \\ & \ddots & \ddots & \\ & & 1 & -1 \\ 0 & & & 1 \end{bmatrix}$$

Then, with $\gamma = -(\alpha+\beta+\delta)$ the product $M_1 M_2$ reads

$$M_1M_2(\lambda) = M_1M_2 = \begin{bmatrix} 1 & -1 & 0 & 0 & & 0 \\ \alpha & \beta & \gamma & \delta & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \alpha & \beta & \gamma & \delta \\ & & 0 & r & s-r & -s \\ 0 & & 0 & 0 & 0 & 1 \end{bmatrix}$$

Except for the first row this matrix realizes the coefficient matrix of (2.1) (multiplied by h) which is

$$\hat{M}(\lambda) = \hat{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 \\ \alpha & \beta & \gamma & \delta & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \alpha & \beta & \gamma & \delta \\ & & 0 & r & s-r & -s \\ 0 & & 0 & 0 & 0 & 1 \end{bmatrix}$$

with

$$r = -\frac{\varepsilon}{h}, \quad s = b + \frac{\varepsilon}{h}$$

and

$$\alpha = -\frac{\varepsilon}{h} + b(\frac{1}{2} - \lambda), \quad \beta = 2\frac{\varepsilon}{h} + 3b\lambda, \quad \gamma = -\frac{\varepsilon}{h} - b(\frac{1}{2} + 3\lambda), \quad \delta = b\lambda.$$

Due to $\lambda \geq 0$ and $b > 0$ it follows $\delta \geq 0$ and $\alpha + \beta = \varepsilon/h + b(2\lambda + 1/2) > 0$. Hence $M_1(\lambda)$ is an M-matrix iff $\alpha \leq 0$. This is equivalent to the condition

$$\lambda \geq \frac{1}{2} - \frac{\varepsilon}{bh}. \tag{2.2}$$

Because M_2 is an M-matrix, too, under the condition (2.2) (and $\lambda \geq 0$) the matrices $M_1(\lambda)$ and M_2 and therefore the matrix $M_1M_2(\lambda)$ are inverse-monotone.
Let us remark, that $M_1M_2(\lambda)$ realizes the coefficient matrix of (2.1) (multiplied by h), if we replace the homogeneous Dirichlet boundary condition $u(x_0) = 0$ by the Neumann boundary condition $u'(x_0) = 0$ and use a common discretization.

For the case $\lambda = \lambda^* = \max\{0, \frac{1}{2} - \frac{\varepsilon}{bh}\}$ in which $\alpha = 0$ the matrix $\hat{M}(\lambda)$ is inverse-monotone, too.
Namely, the relation $\hat{M}(\lambda^*)v \geq 0$ implies $M_1 M_2(\lambda^*)[v_2, v_2, v_3, ..., v_{N+1}]^T \geq 0$. Because $M_1 M_2(\lambda^*)$ is inverse-monotone, we conclude $v_i \geq 0$ for $i = 2, 3, ..., N+1$. Additionally, $v_1 \geq 0$ due to the first inequality of $\hat{M}(\lambda^*)v \geq 0$. Thus, $\hat{M}(\lambda^*)$ is inverse-monotone.
Remark: Numerical experiments leads to the conjecture, that $\hat{M}(\lambda)$ is inverse-monotone for $\lambda > \lambda^*$, too, but in the moment we are not able to prove it.
Moreover, Kopteva [5] verified for $\lambda = 1/2$ the uniform (with respect to ε) boundedness of the related discrete Green's function. This property allows to prove error estimates as well for smooth solutions on a uniform mesh as for solutions with layers on an adapted mesh.
Instead of using a stabilization term of the form $h^2 u'''$ one can also use $h^3 u^{(4)}$. A well known stencil to approximate the fourth-order derivative is given by

$$\frac{u(x_{i-2}) - 4u(x_{i-1}) + 6u(x_i) - 4u(x_{i+1}) + u(x_{i+2})}{h^4} = u^{(4)}(x_i) + 0(h^2).$$

Therefore it is natural to stabilize the central scheme in the interior mesh points by

$$\begin{aligned} &-\varepsilon D^+ D^- u_i - b D_0 u_i \\ &\quad + \frac{b\gamma}{h}(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}) = f_i, \quad i = 2, ..., N-2. \end{aligned} \tag{2.3}$$

Here $\gamma > 0$ is a parameter.
It is clear that the same discretization and therefore also stabilization can only be used for interior mesh points. For the five-point scheme (2.3) in the mesh points x_1 and x_{N-1} some modification is necessary. In the points x_1, x_{N-1} we follow the idea of Frjazinov [9] and choose the stabilization and, consequently, the discretization in such a way that the matrix M corresponding to the stabilization term is symmetric and positive semi-definite. Because the stabilization (2.3) consists of a difference approximation of the fourth-order derivative the following question arises: which results are known for symmetric and positive semi-definite difference schemes for fourth-order differential operators?
It turns out, that the authors of [1] in contrast to many other books discuss this question. Let us consider the boundary value problem

$$(p(x)y'')'' + r(x)y = f(x) \qquad \text{on } (0,1)$$

subject to the boundary conditions

$$\begin{array}{rrclrcl} \text{(i)} & y(x_b) & = & \alpha, & y'(x_b) & = & \beta \\ \text{(ii)} & y(x_b) & = & \alpha, & y''(x_b) & = & \beta \\ \text{(iii)} & y''(x_b) & = & \alpha, & (py'')'(x_b) & = & \beta \end{array}$$

with $x_b = 0$ and $x_b = 1$.

For instance, a discretization of the boundary conditions (i) leads to the following matrix M of format $(N-1)\times(N-1)$ (in the case $p\equiv 1,\ \ r\equiv 0,\ \ f\equiv 0$):

$$M = \begin{bmatrix} 7 & -4 & 1 & & & & 0 \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ 0 & & & & 1 & -4 & 7 \end{bmatrix}$$

With $v=(v_1,...v_{N-1})^T$ the matrix M satisfies [1] (here $(\cdot,\cdot)$ denotes the Euclidean scalar product)

$$(M\,v,v)=\tfrac{1}{2}z_0^2+\sum_{i=1}^{N-1}z_i^2+\tfrac{1}{2}z_N^2, \tag{2.4}$$

here $z_0=-2v_1, z_N=-2v_{N-1},\ \ z_i=-v_{i-1}+2v_i-v_{i+1},\ v_0=v_N=0.$
Alternatively, based on the boundary conditions (ii) the generated matrix M^* is almost identically with M (with the exception that the number 7 is to replace by 5) and satisfies

$$(M^*\,v,v)=\sum_{i=1}^{N-1}z_i^2 \tag{2.5}$$

instead of (2.4).
Thus, we can, for instance, complete the discretization of (1.1) based on (2.3) by

$$\left.\begin{aligned} -\varepsilon D^+D^-u_1-bD_0u_1+\frac{b\gamma}{h}(\tau u_1-4u_2+u_3) &= f_1 \\ -\varepsilon D^+D^-u_i-bD_0u_i & \\ +\frac{b\gamma}{h}(u_{i-2}-4u_{i-1}+6u_i-4u_{i+1}+u_{i+2}) &= f_i, \quad i=2,...,N-2, \\ -\varepsilon D^+D^-u_{N-1}-bD_0u_{N-1} & \\ +\frac{b\gamma}{h}(u_{N-3}-4u_{N-2}+\tau u_{N-1}) &= f_{N-1}, \\ u_0=u_N &= 0 \end{aligned}\right\} \tag{2.6}$$

and call it Frjasinov-type difference scheme. So far the parameter τ admits the values 7 or 5; later we will still generate a scheme with $\tau=6$.
It is obvious that with the property (2.4) of the matrix M we obtain better stability properties of the scheme. While for central differencing we only have (L denotes the difference operator generating the corresponding scheme)

$$(Lu_h,u_h)_{0,h}=\varepsilon|u_h|_1^2,$$

for the scheme (2.6) and $\gamma>0$ we get improved stability due to

$$(Lu_h,u_h)_{0,h}=\varepsilon|u_h|_1^2+b\gamma(M\,u_h,u_h).$$

On the other hand: also with the improved stability of the scheme oscillations of the discrete solution are possible as our numerical experiments show.

In [8] a scheme of type (2.6) for $\gamma = 1/4$ is called weakly monotone because the difference equation

$$a_4 y_{i-2} - a_3 y_{i-1} + a_2 y_i - a_1 y_{i+1} + a_0 y_{i+2} = 0 \tag{2.7}$$

with

$$a_0 = a_4 = \frac{b}{4h}, \quad a_{3,1} = \pm\frac{b}{2h} + \frac{\varepsilon}{h^2} + \frac{b}{h}, \quad a_2 = \frac{2\varepsilon}{h^2} + \frac{3b}{2h}$$

admits the following property: all roots the characteristic equation of (2.7) are real and positive or have a positive real part [8]. It seems that this property excludes wild oscillations but the influence of the discretization in the grid points near to the boundary is so far not absolutely clear.

Further, we do not know error estimates for Frjasinov-type schemes with respect to the maximum norm in the singularly perturbed case.

3 Related schemes generated by stabilizing linear finite elements

The difference scheme (2.1) is based on an $0(h^2)$ perturbation of the third order derivative. In a finite element context, one could realize that perturbation by the discretization: Find $u_h \in V_h$ such that

$$\begin{aligned} \varepsilon({u_h}', {v_h}') - (b{u_h}', v_h) + \frac{b}{2}h\sum_{i=1}^{N-1}[{u_h}']_i(v_h(x_{i-1}) - v_h(x_i)) \\ = (f, v_h) \qquad \forall v_h \in V_h. \end{aligned} \tag{3.1}$$

Here $V_h \subset H_0^1(0,1)$ denotes the space of linear finite elements and $[\cdot]_i$ the jump of a discontinuous function in the point x_i. The scheme generated by (3.1) coincides with (2.1) and $\lambda = \frac{1}{2}$. So far to our best knowledge nobody observed the possibility to generate that scheme based on (3.1) and there does not exist an error analysis for the finite element approach.

It is much more popular to stabilize based on approximations of the fourth-order derivative. Burman and Hensbo introduced in [3] the edge stabilization of the Galerkin method. For problem (1.1) with constant coefficients, the method has the form

$$\varepsilon({u_h}', {v_h}') - (b{u_h}', v_h) + b\,\gamma\,h^2\sum_{i=1}^{N-1}[{u_h}']_i[{v_h}']_i = (f, v_h) \qquad \forall v_h \in V_h. \tag{3.2}$$

It is not difficult to see that (3.2) is equivalent to the difference scheme

$$\begin{aligned} -\varepsilon D^+D^-u_1 - bD_0u_1 + \frac{b\gamma}{h}(5u_1 - 4u_2 + u_3) &= f_1, \\ -\varepsilon D^+D^-u_i - bD_0u_i & \\ +\frac{b\gamma}{h}(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}) &= f_i, \quad i = 2, ..., N-2, \\ u_0 = u_N &= 0 \end{aligned}$$

(we omit it the corresponding equation in x_{N-1}), which is scheme (2.6) with $\tau = 5$.
The method (3.2) belongs to the class of symmetric stabilization methods of the general form

$$a_G(u_h, v_h) + S(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h. \tag{3.3}$$

Here $a_G(.,.)$ denotes the bilinear form of the pure Galerkin approach and $S(\cdot,\cdot)$ the symmetric stabilization. Remember, that in the finite difference framework Frjasinov-type schemes are based on a similar idea.
Introducing

$$\|w\|_E^2 = \varepsilon |w|_1^2 + |w|_0^2 + S(w, w),$$

typical error estimates for symmetric stabilization methods do have the form (for linear finite elements)

$$\|u - u_h\|_E \leq c(\varepsilon^{1/2} h + h^{3/2})|u|_2. \tag{3.4}$$

Because different methods are analyzed in different norms, a fair comparison of different methods is not easy. We simply use the maximum norm in our numerical experiments presented later.
Next we present two variants of projection methods. In the first class of projection methods one uses a projection π into V_h . The discretization is given by

$$a_G(u_h, v_h) + b\gamma h \,\langle {u_h}' - \pi({u_h}'), {v_h}' - \pi({v_h}')\rangle = (f, v_h) \qquad \forall v_h \in V_h. \tag{3.5}$$

Here $\langle .,. \rangle$ denotes some arbitrary scalar product.
Let us introduce the special scalar product

$$\langle w, v \rangle := \sum_{i=1}^{N-1} h \frac{(wv)(x_{i-1}) + (wv)(x_i)}{2}$$

and denote by $\pi w \in V_h$ the projection with respect to that discrete scalar product, i.e. $\pi w \in V_h$ is defined by

$$\langle \pi w, v_h \rangle = \langle w, v_h \rangle \qquad \forall v_h \in V_h.$$

Then, the method (3.5) generates the scheme (again: $u_0 = u_N = 0$ and we omit the equation for $i = N - 1$)

$$\begin{aligned}
-\varepsilon D^+ D^- u_1 - b D_0 u_1 + \frac{b\tilde{\gamma}}{4h}(7u_1 - 4u_2 + u_3) &= f_1, \\
-\varepsilon D^+ D^- u_i - b D_0 u_i & \\
+ \frac{b\tilde{\gamma}}{4h}(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}) &= f_i \quad , i = 2, ..., N-2,
\end{aligned}$$

which is nothing else scheme (2.6) with $\tau = 7$ and $\gamma = \tilde{\gamma}/4$.
Remark: Codina [4] proposed a nonsymmetric variant of (3.5) based on the L_2 scalar product $(\cdot,\cdot)$:

$$a_G(u_h, v_h) + b\gamma h({u_h}' - \pi(u_h'), v_h') = (f, v_h) \qquad \forall v_h \in V_h. \tag{3.6}$$

In his original version the L_2-projection is used (which is practically bad). Then the symmetric and nonsymmetric version coincide because

$$({u_h}' - \pi(u_h'), \pi(v_h')) = 0.$$

But if we replace π in (3.6) by some other local projection and use the Oswald projector (or the Clemént projector), for instance, the symmetric and the nonsymmetric version are different. It turns out that we generate a seven-point difference scheme with the symmetrized version but the difference scheme (2.6) when using the nonsymmetric version (3.6). ☐

The second class of symmetric projection methods uses a macro mesh $\mathcal{T}_{\mathcal{M}}$ and a second finite element space M_h of possibly discontinuous finite elements. Now the projector π_h projects into M_h and the method reads

$$a_G(u_h, v_h) + b\gamma h \sum_M ({u_h}' - \pi_h({u_h}'), {v_h}' - \pi_h(v_h'))_M = (f, v_h) \quad \forall v_h \in V_h. \tag{3.7}$$

We denote by $(.,.)_M$ the L_2 scalar product restricted to some $M \in \mathcal{T}_{\mathcal{M}}$. For linear elements it is standard to choose M_h to be the space of piecewise constants on the macro mesh and to define the projection as the piecewise L_2-projection (we do not discuss schemes based on enrichment of approximation spaces, see [6]).

Often a $2h$-mesh is proposed to be the macro mesh. Then the following stencils are generated by the stabilization term:

$$\left.\begin{array}{rcll}
-\varepsilon D^+D^-u_i - bD_0u_i + \dfrac{b\gamma}{h}(-u_{i-1} + 2u_i - u_{i+1}) & = & f_i, & \text{if } i \text{ is odd,} \\
-\varepsilon D^+D^-u_i - bD_0u_i & & & \\
+\dfrac{b\gamma}{2h}(u_{i-2} - 2u_{i-1} + 2u_i - 2u_{i+1} + u_{i+2}) & = & f_i, & \text{if } i \text{ is even,} \\
u_0 = u_N & = & 0 &
\end{array}\right\} \tag{3.8}$$

That means: the stencils generated are comparable with (1.4) and therefore only a $0(h)$ perturbation of the second order derivative. Therefore, this variant of the local projection method is not to recommend in comparison to the methods discussed so far.

It seems to be better to use the piecewise constant projection onto the Voronoi boxes related to x_i. This method generates in the interior mesh points not close to the boundary again the stencil $\{1, -4, 6, -4, 1\}$, but in the points near the boundary, for instance x_1,

$$-\varepsilon D^+D^-u_1 - bD_0u_1 + \frac{b\gamma}{4h}(6u_1 - 4u_2 + u_3) = f_1.$$

4 How to choose the parameters?

If N is even, for central differencing it is known that $\lim\limits_{\varepsilon \to 0} u_1 = \infty$, thus we have unrealistic, even unbounded oscillations. Any stabilization method should avoid oscillations or at least guarantee that occurring oscillations are small.

For the scheme (2.1) we have inverse-monotonicity if $\lambda \geq \max\{0, \frac{1}{2} - \frac{\varepsilon}{bh}\}$. Most of the schemes mentioned (except the scheme related to (3.8)) are weakly monotone in the sense of Stoyan.

Because the behavior of the discrete solution near the layer is very important let us look on the discrete equation in the first mesh point. In all cases (except(3.8)) it has the form

$$\frac{2\varepsilon}{h}u_1 - \frac{\varepsilon}{h}u_2 - b\frac{u_2}{2} + b\,\mu(Au_1 - Bu_2 + u_3) = f_1 h$$

with constants A, B characterizing the scheme and some parameter μ. Because for $\varepsilon \ll h$ the first mesh point has already some distance to the layer, $u_2 = u_3 = U$ should imply $u_1 = U$. This leads to (assuming h to be small)

$$\mu = \frac{\frac{1}{2} - \frac{\varepsilon}{bh}}{A - B + 1}. \tag{4.1}$$

If we neglect $\varepsilon/(bh)$ this gives $\lambda = \frac{1}{2}$ for scheme (2.1) and $\gamma = \frac{1}{4}$ for scheme (2.6), for instance. But the numerical experiments of the next Section will show that the optimal choice of the parameters is very important.

Equivalently to the above derivation of (4.1), we can write the scheme in the first mesh point in the form

$$-\frac{\varepsilon}{h}(u_0 - 2u_1 + u_2) - b\frac{u_2 - u_0}{2} + b\,\mu\,[(B - A - 1)u_0 + Au_1 - Bu_2 + u_3] = f_1 h$$

and require that u_0 (if different from zero) has no influence on the other u_i, $i = 1, 2, \ldots$. That gives

$$-\frac{\varepsilon}{h} + b/2 + b\mu(B - A - 1) = 0.$$

and therefore (4.1), too.

Analogously, considering for five-point schemes like (2.3) in the second mesh point, it follows that it is necessary to switch off the stabilization. That is verified in our experiments, too.

Remark: Our approach corresponds to a necessary convergence condition for $h \to 0$ and $\varepsilon \ll h$ for a problem without a layer component in the solution decomposition. It is also possible in the usual way [7] to derive a necessary condition for uniform convergence of u_1 towards $u(x_1)$ but this results in a complicated formulae for $\mu = \mu(q)$ with $q = (bh)/(2\varepsilon)$. Because exponential fitting in 2D, especially for problems with characteristic layers, is useless we do not follow this approach. □

5 Numerical experiments

In our numerical experiments we set $b = 1$ and study

$$-\,\varepsilon\,u'' - u' = f(x), \quad x \in (0, 1), \qquad u(0) = u(1) = 0$$

for equidistant meshes characterized by $N = N_i = 10 \cdot 2^{i-1}$, $h_i = \dfrac{1}{N_i}$, $i = 1, 2, ..., 20$, in general for $\varepsilon = 10^{-5}$, and present all results in the max-norm and in dependence of N.
First we verified for a problem without layers and a smooth solution the convergence rate of the four-point scheme (2.1) and the five-point scheme (2.6). We fixed f in such a way that $u(x) = \sin(\pi x)$ solves the problem.

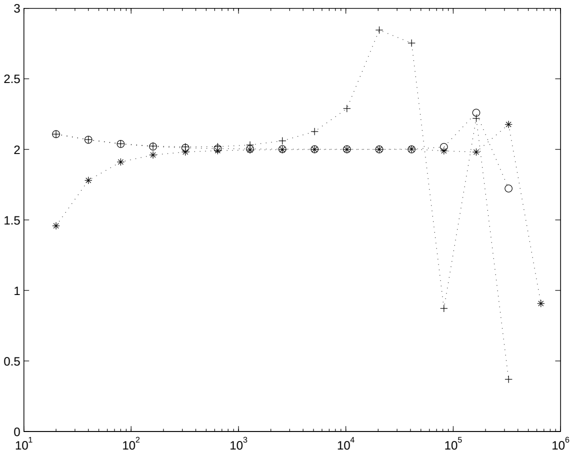

Figure 1: Order of convergence for the scheme (2.1) with
$\circ$ ... $\lambda = 1/2$, $*$ ... $\lambda = 2$,
$+$ ... $\lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$

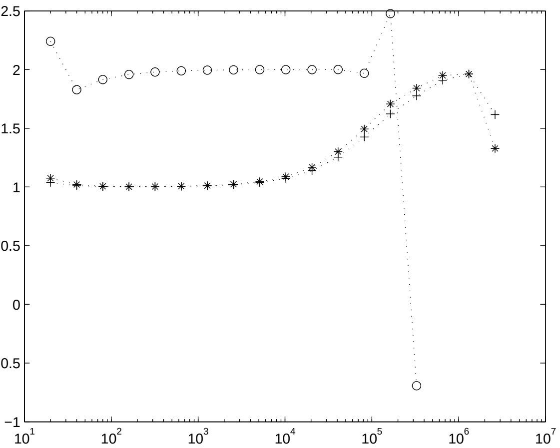

Figure 2: Order of convergence for the five-point schemes (2.6) with
$\circ$ ... $\tau = 5$, $*$ ... $\tau = 6$,
$+$ ... $\tau = 7$

Figure 1 shows convergence of order 2 for the four-point scheme (2.1) for different values of λ such that theory and experiment coincide.
For the five-point schemes (2.6) we consider $\gamma = 1/4$ and the cases $\tau = 5, 6, 7$. If $\tau = 5$ we have consistency of order 1 in the mesh points close to the boundary, in the other cases this order is 0. We hope for convergence of order 2 (but do not have a proof in the maximum norm), but only the scheme with $\tau = 5$ clearly shows this behaviour, see Figure 2.
This a little surprising fact can be explained with the consistency order near to the boundary: in the non-singularly perturbed case for k-th order equations consistency of order m in the interior and $m - k$ near to the boundary gives convergence of order m, see [2]. But for singularly perturbed problems this property does not hold uniformly with respect to ε.
Remark that in Figure 1 and 2 we stop the output for some value of N because for larger N the influence of roundoff error is dominant.
To study the numerical behaviour of our schemes for a problem with a layer we choose $f(x) = \mathrm{e}^{x-1}$ and obtain an exact solution of the structure

$$u(x) = C_1 - \frac{1}{1+\varepsilon}\,\mathrm{e}^{1-x} + C_2\,\mathrm{e}^{-x/\varepsilon}\,.$$

Because we want to study equidistant meshes it makes no sense to measure convergence rates. Instead, we observe the error behaviour for fixed small ε and decreasing h.

It is well known that for upwind schemes the error at the layer can increase for decreasing h in a certain region depending on ε, see [7], Chapter I, figure 2.1. We expect the same principal behaviour for our schemes but want to study this effect and its dependence on the parameters λ and γ in the schemes.

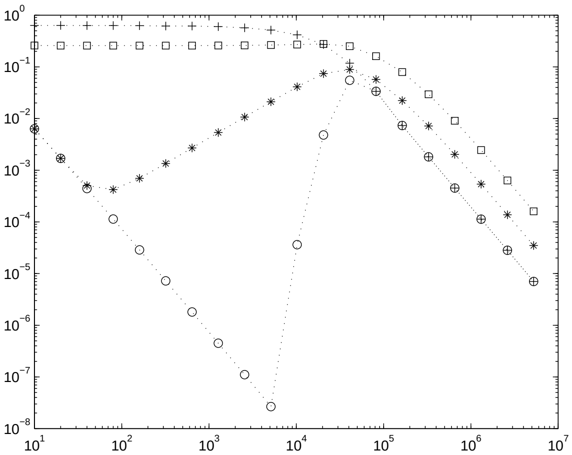

Figure 3: Error for the four-point scheme (2.1) with $+$ ... $\lambda = 0$, $*$ ... $\lambda = 1/2$, $\square$ ... $\lambda = 2$, $\circ$ ... $\lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$

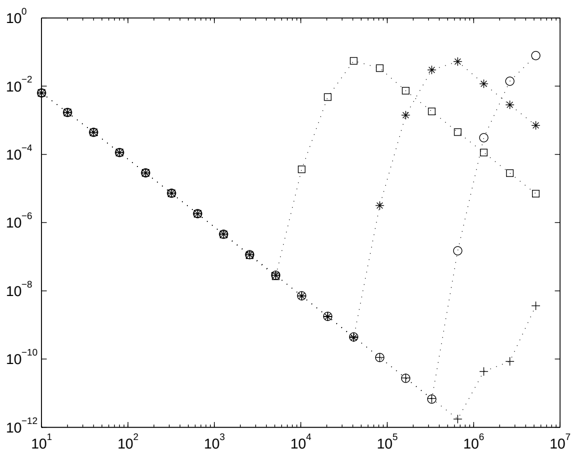

Figure 4: Error for the four-point scheme (2.1) in dependence of $\square$ ... $\varepsilon = 10^{-5}$, $*$ ... $\varepsilon = 10^{-6}$, $\circ$ ... $\varepsilon = 10^{-7}$, $+$ ... $\varepsilon = 10^{-8}$

Figure 3 shows the error behaviour for the four-point scheme (2.1) for different values of λ.

It turns out that the choice $\lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$ is the best, theoretically to explain by the fact that for this choice the scheme does not employ the "outflow" boundary value u_0.

Figure 4 shows the results for different ε.

For the five-point scheme (2.6) (now we always use the variant with $\tau = 5$) the choice of the parameter γ is extremely important. If we simply choose some positive value, say $\gamma = 1/4$, the result for $\varepsilon \ll h$ is bad. Figure 5 shows a comparison with the four-point scheme (2.1) for $\lambda = 1/2$ and with one-point upwinding (central differencing with upwinding only in the nearest mesh point to the layer).

To eliminate the influence of u_0, it is necessary to choose in the first mesh point $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$, but additionally $\gamma_2 = 0$ in the second mesh point! Figure 6 clearly demonstrates that the choice $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$ alone is not sufficient.

If $\gamma_1 = \max\{0, 1/2 - \varepsilon/(bh)\}$ and $\gamma_2 = 0$, the choice of the stabilization parameter in the remaining mesh points is not important.

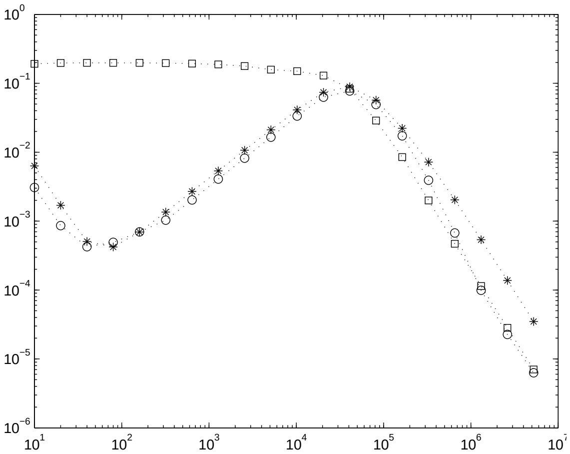

Figure 5: Error for
○ ... one-point upwind scheme (1.2),
$*$ ... four-point scheme (2.1) with $\lambda = 1/2$,
□ ... five-point scheme (2.6) with $\gamma = 1/4$

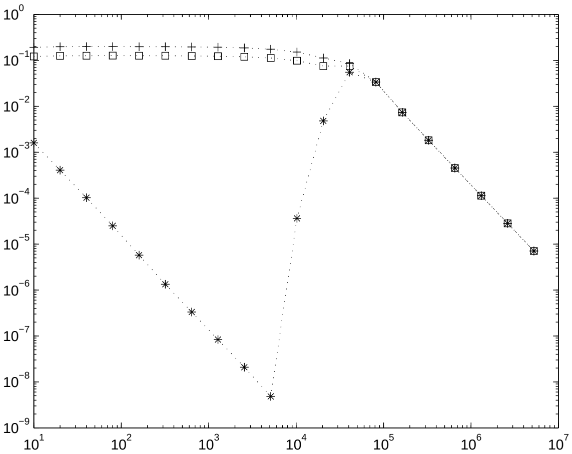

Figure 6: Error for the five-point scheme (2.6) with $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$ and
$+$ ... $\gamma_2 = \gamma_1$, $*$ ... $\gamma_2 = 0$,
□ ... $\gamma_2 = \max\{0, (1/2 - \varepsilon/(bh))/3\}$

Figure 7 shows that the four-point scheme and the five-point scheme yield similar results if the parameters are chosen in an optimal way, i.e. for the scheme (2.1) $\lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$ and for the scheme (2.6) $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$ and $\gamma_2 = 0$.

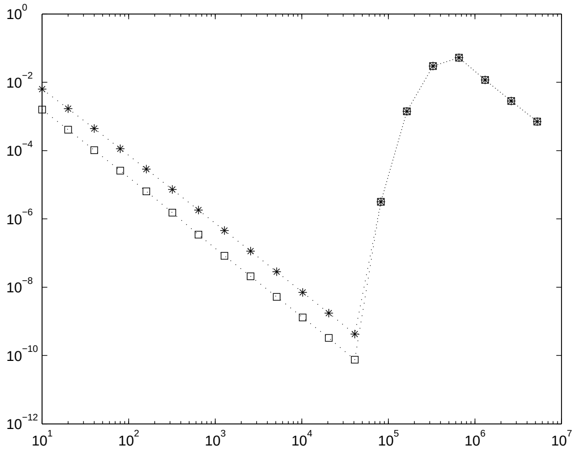

Figure 7: Error for optimal parameters and
$*$ ... scheme (2.1) , □ ... scheme (2.6)

Finally, we still study scheme (3.8). If we do not choose $\gamma_1 = \max\{0, (1/2-\varepsilon/(bh))/2\}$ and $\gamma_2 = 0$, the scheme is similarly bad as other five-point schemes. But for $\gamma = \max\{0, (1/2 - \varepsilon/(bh))/2\}$ for all odd i, $\gamma = 0$ for $i = 2$ and $\gamma = 1/4$ for all other even i the scheme is not so good as the schemes (2.1) and (2.6) for optimal parameters, compare Figure 7 and Figure 8.

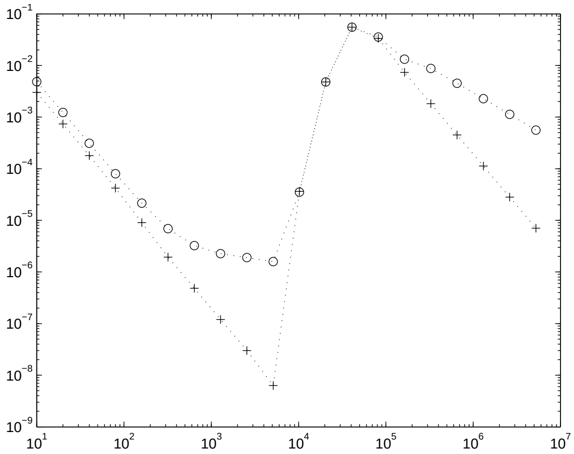

Figure 8: Error for the scheme (3.8) and
+ ... $\gamma = \max\{0, (1/2 - \varepsilon/(bh))/2\}$ for $i = 1$, $\gamma = 0$ for all other i,
○ ... $\gamma = \max\{0, (1/2 - \varepsilon/(bh))/2\}$ for all odd i, $\gamma = 0$ for $i = 2$ and $\gamma = 1/4$ for all other even i

Summarizing we observed that the schemes (2.1) and (2.6) only beat the one-point upwind scheme if the stabilization parameters are extremely careful chosen, especially for the five-point scheme. Of course, in 2D it is much more complicated to tune the parameters in such a way that the outflow boundary values do not influence the numerical solution. This is an important question for further research.

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