Stochastic semigroups: their construction by perturbation and approximation

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Abstract. The main object of the paper is to present a criterion for the minimal semigroup associated with the Kolmogorov differential equations to be stochastic in $\ell^1$. Our criterion uses a weighted $\ell^1$-space. As an abstract preparation we present a perturbation theorem for substochastic semigroups which generalizes known results to the case of ordered Banach spaces which need not be $AL$-spaces. We also consider extensions of Kolmogorov’s equations to spaces of measures. In an appendix we present a version of the Miyadera perturbation theorem for positive semigroups on ordered Banach spaces.

Key words: semigroup, substochastic, stochastic, perturbation, approximation, Kolmogorov differential equations

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1 Introduction

Let $X$ be an ordered real Banach space with a generating (reproducing) cone $X_+$, $X = X_+ - X_+$, and a norm which is additive on $X_+$, $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X_+$.

A $C_0$-semigroup $(S(t); t \geq 0)$ (of bounded linear operators) on $X$ is called substochastic (stochastic) if $S(t)$ is positive and $\|S(t)x\| \leq \|x\|$ ($\|S(t)x\| = \|x\|$) for all $x \in X_+$, $t \geq 0$.

The norm on $X_+$ can be uniquely extended to a positive bounded linear functional $\varphi \in X^*$ satisfying $\varphi x = \|x\|$ for all $x \in X_+$. The generator $A$ of a stochastic semigroup necessarily satisfies $\varphi Ax = 0$ for all $x \in D(A) \cap X_+$.

It is the aim of this note to indicate conditions for a $C_0$-semigroup defined by a limiting procedure to be stochastic. Differently from [17], [25; Sec. 2], [3] we work with an auxiliary Banach space $X_1$ that is continuously and densely embedded into $X$. This space may be of its own interest: in case of the Kolmogorov differential equations (Section 4), it can be chosen as the first moment space $\ell^{11} = \{x = (x_j)_{j=0}^\infty; \|x\|_1 := \sum_{j=0}^\infty (1 + j)|x_j| < \infty\}$. By perturbation (Section 2) we will construct substochastic $C_0$-semigroups on $X$ which leave the auxiliary Banach space invariant and induce $C_0$-semigroups thereon. These semigroups can also be obtained by an approximation procedure (Section 3). As a second application we present a measure-valued generalization of Kolmogorov’s differential equations (Section 5) whose solutions are associated with Markov transition functions. In an appendix we present a version of the Miyadera perturbation theorem for positive semigroups on ordered Banach spaces. This result is needed in Section 2.

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2 Perturbation

A generating cone $X_+$ is always non-flat; cf. Remark A.1(a). The additivity of the norm on $X_+$ implies that the the norm is monotone, and hence $X_+$ is normal; cf. [10; Proposition A.2.2]. The additivity of the norm on $X_+$ also implies that any bounded monotone sequence is convergent.

2.1 Remarks. (a) Let $S$ be a positive $C_0$-semigroup on $X$, with generator $A$. We recall that then $D(A)_+ (= D(A) \cap X_+)$ is dense in $X_+$. Moreover, $S$ is substochastic (stochastic) if and only if $\varphi Ax \leqslant 0$ ($\varphi Ax = 0$) for all $x \in D(A)_+$. This follow immediately from the fact that, for $x \in D(A)_+$, one has

$$\frac{d}{dt} \|S(t)x\| = \frac{d}{dt} \varphi(S(t)x) = \varphi(AS(t)x).$$

Further, if $D$ is a core for $A$, and $\varphi Ax = 0$ for all $x \in D$, then it follows that $\varphi Ax = 0$ for all $x \in D(A)$, and that therefore $S$ is stochastic.

(b) Note that, although the norm of $X$ is supposed to be additive on $X_+$, we do not suppose $X$ to be an $AL$-space. In fact, the order need not be associated with a Banach lattice structure. We want to present the abstract part of the paper in this general setting, although in the applications presented in Sections 4 and 5 the space $X$ will be an $AL$-space. Nevertheless, we think that it is interesting to know that certain properties are valid in the general setting we present here.

2.2 Theorem. Let $S_0$ be a positive $C_0$-semigroup on $X$, with generator $A_0$. Let $B : D(A_0) \to X$ be a positive linear operator satisfying

$$\varphi(A_0 + B)x \leqslant 0 \quad (x \in D(A_0)_+).$$

(a) Then, for all $x \in D(A_0)_+$, one has

$$\int_0^\infty \|BS_0(s)x\| \, ds \leqslant \|x\|.$$

For all $0 \leqslant r < 1$, the operator $A_0 + rB$ is the generator of a substochastic $C_0$-semigroup $S_r$. For $0 \leqslant r \leqslant r' < 0$ one has $S_r(t) \leqslant S_{r'}(t)$ ($t \geqslant 0$).

(b) $S(t) := s\text{-lim}_{r \to 1} S_r(t)$ exists, uniformly for $t$ in bounded subsets of $[0, \infty)$. $S$ thus defined is a substochastic $C_0$-semigroup on $X$. The generator of $S$ is an extension of $A_0 + B$.

(c) $S$ is the smallest positive semigroup (also called the minimal semigroup) whose generator is an extension of $A_0 + B$.

Proof. (a) Since $B : D(A_0) \to X$ is positive we obtain that $B(\lambda - A_0)^{-1}$ is positive, hence bounded (cf. [10; Proposition A.2.11]), i.e., $B$ is $A_0$-bounded. Moreover, for $0 \leqslant t < \infty$, we obtain

$$\int_0^t \|BS_0(s)x\| \, ds = \int_0^t \varphi(BS_0(s)x) \, ds$$

$$\leqslant - \int_0^t \varphi(A_0S_0(s)x) \, ds = -\varphi \int_0^t A_0S_0(s)x \, ds$$

$$= \varphi(x - S_0(t)x) = \|x\| - \|S_0(t)x\| \leqslant \|x\|$$
and hence
\[ \int_0^\infty \| B S_0(s)x \| \, ds \leq \| x \|, \]
for all \( x \in D(A_0)_+ \) (cf. [25; Lemma 1.2(a)]).

Using Theorem A.2, one proves the remaining assertions in the same way as in [25; Lemma 1.2 and Proposition 1.4].

2.3 Remark. The first version of a result as in Theorem 2.2 is due to Kato [17]. The application of the Miyadera perturbation theorem in this context goes back to [25]. Further developments can be found in [3], [15], [7], [6].

Next, we show a result on invariant subspaces of the dual semigroup. This result will be used in Section 5.

2.4 Proposition. Let the assumptions of Theorem 2.3 be satisfied, and let be a weakly * sequentially closed subspace of \( X^* \) which is invariant under \( ((\lambda - A_0)^{-1})^* \) and \( (B(\lambda - A_0)^{-1})^* \). Then \( Y \) is invariant under the semigroups \( S_r^* \) and \( S^* \), with \( S_r \) and \( S \) from Theorem 2.3.

Proof. It follows from the proof of Theorem A.2 that the spectral radius of \( F(\lambda) = B(\lambda - A_0)^{-1} \) does not exceed one. Set \( A_r = A_0 + rB \). Then
\[
(\lambda - A_r)^{-1} = (\lambda - A_0)^{-1} \sum_{j=0}^\infty (rF(\lambda))^j.
\]
Since \( Y \) is weakly * sequentially closed and invariant under \( ((\lambda - A_0)^{-1})^* \) and \( F(\lambda)^* \), it is invariant under \( ((\lambda - A_r)^{-1})^* \). Since \( S_r(t) = s\text{-}\lim_{n \to \infty} (I - (t/n)A_r)^{-n} \), \( Y \) is invariant under \( S_r^*(t) \), and also under \( S^*(t) \) because \( S(t) \) is the strong limit of \( S_r(t) \). □

If, in Theorem 2.2, one assumes \( \varphi(A_0 + B)x = 0 \) for all \( x \in D(A_0) \cap X_+ \), one cannot conclude that \( S \) is stochastic, in general; cf. [17; §4, Example 3]. Discussing conditions for this to hold has been a major objective in most of the pertinent work; cf. [17], [21], [25], [3], [8], [5], [4], [2], [15], [6]. Besides the development of the theory in the more general context, it was the main motivation of the present paper to indicate a new type of sufficient condition; cf. Theorem 2.7.

2.5 Assumption. Let \( X_1 \) be a subspace of \( X \) such that the following hold:

- There exists a norm \( \| \cdot \|_1 \) on \( X_1 \) which makes it a Banach space.
- \( (X_1, \| \cdot \|_1) \) is continuously embedded into \( (X, \| \cdot \|) \) and \( X_1 \cap X_+ \) is dense in \( X_+ \).
- \( X_{1,+} := X_1 \cap X_+ \) is a generating cone for \( X_1 \) and \( \| \cdot \|_1 \) is additive on \( X_{1,+} \).

It follows from these assumptions that there exists a unique positive functional \( \varphi_1 \in X_1^* \) which coincides with \( \| \cdot \|_1 \) on \( X_{1,+} \).

2.6 Proposition. Let the assumptions of Theorem 2.2 be satisfied. Assume that \( X_1 \) is a subspace \( X \) which satisfies Assumption 2.5.
Assume that \( S_0 \) induces a (necessarily positive) \( C_0 \)-semigroup \( \hat{S}_0 \) on \( (X_1, \| \cdot \|_1) \). Let \( \hat{A}_0 \) denote its generator. (Note that \( \hat{A}_0 \) is the restriction of \( A_0 \) to \( D(\hat{A}_0) = \{ x \in D(\hat{A}_0) \cap X_1 : A_0 x \in X_1 \} \).) Assume additionally that \( B(D(\hat{A}_0)) \subseteq X_1 \), and assume that there is a constant \( c > 0 \) such that

\[
\varphi_1(A_0 + B)x \leq c\|x\|_1 \quad (x \in D(\hat{A}_0)_+) .
\]

Then the semigroup \( S \) from Theorem 2.2 leaves \( X_1 \) invariant and induces a positive \( C_0 \)-semigroup \( \hat{S} \) on \( X_1 \). \( \hat{S} \) is the smallest positive \( C_0 \)-semigroup on \( X_1 \) whose generator is an extension of \( \hat{A}_0 + B \). For \( x \in X_{1+}, t \geq 0 \), we have \( \|S(t)x\| \leq e^{ct}\|x\| \), i.e., the rescaled semigroup \( (e^{-ct}\hat{S}(t))_{t \geq 0} \) is substochastic on \( X_1 \).

**Proof.** Note first that the hypothesis can be reformulated as

\[
\varphi_1((\hat{A}_0 - c) + \hat{B})x \leq 0, \quad (x \in D(\hat{A}_0)_+) ,
\]

where \( \hat{B} \) denotes the restriction of \( B \) to \( D(\hat{A}_0) \). Therefore Theorem 2.2 can be applied to \( X_1 \) and the restricted operators.

Let \( 0 \leq r < 1 \). Then Theorem 2.2(b) implies that \( \hat{A}_0 - c + r\hat{B} \) is the generator of a substochastic semigroup on \( X_1 \), or equivalently, that \( \hat{A}_0 + t\hat{B} \) is the generator of a positive \( C_0 \)-semigroup \( \hat{S}_t \) on \( X_1 \) satisfying \( \|\hat{S}_t(t)x\| \leq e^{ct}\|x\| \) for all \( x \in X_{1+}, t \geq 0 \). It is easy to see that \( \hat{S}_r \) and \( S_r \) (from Theorem 2.2) coincide on \( X_1 \). Taking \( r \to 1 \) one obtains the assertion. \( \square \)

**2.7 Theorem.** Let the assumptions of Proposition 2.6 be satisfied. Assume additionally that \( -A_0 \) is positive and that there exists \( \varepsilon > 0 \) such that (with \( c > 0 \) from Proposition 2.6)

\[
\varphi_1(A_0 + B)x \leq c\|x\|_1 - \varepsilon\|A_0 x\| \quad (x \in D(\hat{A}_0)_+) ,
\]

i.e., for all \( x \in D(\hat{A}_0)_+ \cap X_1 \) with \( A_0 x \in X_1 \).

Then the generator \( \hat{A} \) of \( \hat{S} \) is a restriction of \( A_0 + B \), and the generator \( A \) of \( S \) is the closure of \( \hat{A} \) and of \( A_0 + B \) in \( X \).

If \( \varphi(A_0 + B)x = 0 \) for all \( x \in D(\hat{A}_0)_+ \) then \( S \) is stochastic.

**2.8 Remark.** The assumption that \( -A_0 \) is a positive operator can be replaced by the following:

If \( (x_n) \) is sequence in \( D(\hat{A}_0)_+ \), \( x_n \uparrow x \) in \( X_1 \) and \( \sup_{n \in \mathbb{N}} \|A_0 x_n\| < \infty \), then \( x \in D(\hat{A}_0) \).

**Proof of Theorem 2.7.** Let \( (r_n) \) be a sequence in \( [0, 1] \), \( r_n \uparrow 1 \). We shall use the notation \( A_n := A_0 + r_n B \) \((n \in \mathbb{N})\).

We know that \( A_0 + B \subseteq A \) and \( \hat{A} \subseteq A \). In order to show the first assertion we thus have to show \( D(\hat{A}) \subseteq D(\hat{A}_0) \).

Let \( x \in X_{1+}, \lambda > \max\{0, c\} \). Then \( x_n := (\lambda - A_n)^{-1}x \in D(\hat{A}_0)_+ \) \((n \in \mathbb{N})\). By hypothesis,

\[
\varphi_1(-x + \lambda x_n) = \varphi_1 A_n x_n \leq \varphi_1(A_0 + B)x_n \leq c\|x_n\|_1 - \varepsilon\|A_0 x_n\| ,
\]

\[
\varepsilon\|A_0 x_n\| \leq (\lambda - c)\|x_n\|_1 + \varepsilon\|A_0 x_n\| \leq \varphi_1 x = \|x\|_1 .
\]

The sequence \( (x_n) \) is increasing and converges to \( x_1 \) to \( (\lambda - A)^{-1}x \). Since \( A_0 \) preserves monotone sequences, \( (A_0 x_n) \) is a bounded monotone sequence in \( X \) and thus has a limit in \( X \). Since \( A_0 \) is a closed operator, \( (\lambda - A)^{-1}x \in D(A_0) \). This implies \( D(\hat{A}) = (\lambda - A)^{-1}X_1 \subseteq D(A_0) \) since \( X_{1+} \) is a generating cone for \( X_1 \).

The domain \( D(\hat{A}) \) of \( \hat{A} \) is a subset of \( D(A) \), invariant under \( \hat{S} \) (\( = \hat{S} \) on \( X_1 \)) and dense in \( X \), and therefore a core for \( A \) (cf. [20; Theorem X.49]). From \( \hat{A} \subseteq A_0 + B \subseteq A \) we also obtain \( A = \overline{A_0 + B} \).

Now the last statement is a consequence of Remark 2.1(a). \( \square \)
3 Approximation

3.1 Proposition. Let $S_0$, $A_0$, $B$, $S$ be as in Theorem 2.2, in particular recall
\[ \varphi(A_0 + B)x \leq 0 \quad (x \in D(A_0)_+). \]
For $n \in \mathbb{N}$ let $B_n : D(A_0) \to X$ be a positive linear operator, $B_n x \leq Bx$ ($x \in X_+$, $n \in \mathbb{N}$), $B_n x \to Bx$ ($n \to \infty$) for all $x \in D(A_0)$. For $n \in \mathbb{N}$ let $S_n$ be the smallest positive (substochastic) $C_0$-semigroup whose generator is an extension of $A_0 + B_n$ (see Theorem 2.2(a)).

(a) Then $S_n(t) \leq S(t)$ ($n \in \mathbb{N}$) and $S(t) = \lim_{n \to \infty} S_n(t)$ ($t \geq 0$).

(b) Assume additionally that $X_1$ is a subspace of $X$ satisfying Assumption 2.5. As in Proposition 2.6, assume that $S_0$ induces a $C_0$-semigroup $\tilde{S}_0$ on $X_1$, and let $\tilde{A}_0$ denote its generator. Assume that $B_n(D(A_0)) \subseteq X_1$ ($n \in \mathbb{N}$), $B(D(A_0)) \subseteq X_1$, that $B_n x \to Bx$ in $X_1$ ($n \to \infty$) for all $x \in D(\tilde{A}_0)$, and that there is a constant $c > 0$ such that
\[ \varphi_1(A_0 + B)x \leq c \|x\|_1 \quad (x \in D(\tilde{A}_0)_+). \]
Then $S_n$ ($n \in \mathbb{N}$) and $S$ induce positive $C_0$-semigroups $\tilde{S}_n$ ($n \in \mathbb{N}$) and $\tilde{S}$ on $X_1$, respectively, and $\tilde{S}(t) = \lim_{n \to \infty} \tilde{S}_n(t)$ in $X_1$ ($t \geq 0$).

Proof. (a) (cf. [25; proof of Proposition 1.6]) For $0 \leq r < 1$ we obtain from Theorem 2.2(a) that the operators $A_0 + rB_n$ ($n \in \mathbb{N}$) and $A_0 + rB$ are generators of substochastic $C_0$-semigroups $S_{n,r}$ ($n \in \mathbb{N}$) and $S_r$, respectively. The inequalities required in the hypothesis imply $S_{n,r}(t) \leq S_r(t)$ ($t \geq 0$, $n \in \mathbb{N}$). Taking $r \to 1$ we obtain $S_n(t) \leq S(t)$ ($t \geq 0$, $n \in \mathbb{N}$). A minor adaptation of [24; Theorem 1.4] to our context shows that $S_{n,r}(t) \to S_r(t)$ strongly ($n \to \infty$).

Let $x \in X_+$. The inequalities
\[ 0 \leq \| S(t)x - S_n(t)x \| = \| S(t)x - S_r(t)x + (S_r(t)x - S_{n,r}(t)x) + (S_{n,r}(t)x - S_n(t)x) \| \leq \| S(t)x - S_r(t)x \| + \| (S_r(t)x - S_{n,r}(t)x) \| \]
implies $\| S(t)x - S_n(t)x \| \leq \| S(t)x - S_r(t)x \| + \| S_r(t)x - S_{n,r}(t)x \|$. Choosing first $r$ close enough to 1 and then $n$ large enough we can make the right hand side as small as we want. This implies that $S_n(t) \to S(t)$ strongly.

(b) Proposition 2.6 implies that the semigroups $S_n$ ($n \in \mathbb{N}$) and $S$ induce $C_0$-semigroups $\tilde{S}_n$ ($n \in \mathbb{N}$) and $\tilde{S}$ on $X_1$, and that these semigroups are the smallest positive semigroups on $X_1$, whose generators are extensions of $\tilde{A}_0 + \tilde{B}_n$ ($n \in \mathbb{N}$) and $\tilde{A}_0 + \tilde{B}$, respectively. Part (a), applied to these semigroups, yields $\tilde{S}(t) = \lim_{n \to \infty} \tilde{S}_n(t)$ ($t \geq 0$).

3.2 Remark. Assuming monotonicity, i.e., $B_n x \leq B_{n+1} x$ ($x \in D(A)_+$, $n \in \mathbb{N}$), in Proposition 3.1, one could simplify the hypothesis in part (b). In this case the convergence $B_n x \to Bx$ in $X_1$ would follow from the remaining hypotheses.

4 Example: Kolmogorov’s differential equations

The infinite system of differential equations
\[ x_j' = \sum_{k=0}^{\infty} a_{jk} x_k \quad (j = 0, 1, 2, \ldots) \quad (4.1) \]
is known as Kolmogorov’s differential equations provided the coefficients $a_{jk}$ form a Kolmogorov matrix [16; Sec. 23.12], i.e.,
for all $j, k \in \mathbb{N}_0$, $\alpha_{jk} \geq 0$ if $j \neq k$, $\alpha_{jj} \leq 0$,

- $\sum_{j=0}^{\infty} \alpha_{jk} = 0$ for all $k \in \mathbb{N}_0$.

$x_j$ can be interpreted as the probability that the number of individuals in a population is $j$. As we allow the population to go extinct and be possibly resurrected by immigration, we choose the non-negative integers $\mathbb{N}_0$ as state space.

For $j \neq k$, $\alpha_{jk}$ is the rate at which the population size changes from $k$ to $j$, while $-\alpha_{jj}$ is the rate at which a population of size $j$ changes to a size different from $j$.

Typically Kolmogorov's differential equations are considered on the standard sequence space $\ell^1 = \{ x = (x_j)_{j=0}^{\infty}; \| x \| < \infty \}$ with $\| x \| = \sum_{j=0}^{\infty} |x_j|$. See [17], [16; Chap. 23], [13; XVII.9], [14; XIV.7], [25], and the literature cited there. As $\sum_{j=1}^{\infty} jx_j$ is the expected population size, the first moment space $\ell^{11} = \{ x = (x_j)_{j=0}^{\infty}; \| x \|_1 < \infty \}$ with $\| x \|_1 = \sum_{j=0}^{\infty} (1+j)x_j < \infty$ is also a meaningful state space. Interestingly enough, this will help us find a condition for the semigroup associated with (4.1) to be stochastic on $\ell^1$.

4.1 **Assumption.** There exist constants $c, \varepsilon > 0$ such that

$$\sum_{j=0}^{\infty} j\alpha_{jk} \leq c(1+k) - \varepsilon |\alpha_{kk}| \quad \text{for all } k \in \mathbb{N}_0.$$  

Notice that $\sum_{j=0}^{\infty} j\alpha_{jk}$ can be interpreted as expected population growth rate at population size $k$. Let

$$D_0 = \{ x = (x_j) \in \ell^1; \sum_{j=0}^{\infty} |\alpha_{jj}|x_j| < \infty \}.$$  

4.2 **Theorem.** Let $(\alpha_{jk})$ be a Kolmogorov matrix which satisfies Assumption 4.1. Then the closure of the operator $A_\infty: D_0 \to \ell^1$,

$$(A_\infty x)_j = \sum_{k=0}^{\infty} \alpha_{jk} x_k \quad (x = (x_k) \in D_0),$$

is the generator of a stochastic semigroup $S$ on $\ell^1$.

The semigroup $S$ leaves $\ell^{11}$ invariant and induces a strongly continuous semigroup $\tilde{S}$ on $\ell^{11}$. The generator of $\tilde{S}$ is the restriction of $A_\infty$ to

$$D(\tilde{A}) = \{ x \in \ell^{11} \cap D_0; A_\infty x \in \ell^{11} \}.$$  

Moreover $\| S(t)x \|_1 \leq e^{c t}\| x \|_1$ for all $x \in \ell^{11}, t \geq 0$.

This theorem is a consequence of Theorem 2.2, Proposition 2.6 and Theorem 2.7. It is useful, however, also to apply Proposition 3.1 because this provides an approximation result which allows to find conditions under which the semigroup is bounded on $\ell^{11}$ and has a strictly negative essential growth bound [19]. Let

$$\alpha_{[n]}_{jk} = \begin{cases} 
\alpha_{jj} & \text{if } j = k, \\
\alpha_{jk} & \text{if } j, k < n, j \neq k, \\
0 & \text{otherwise} 
\end{cases} \quad (j, k, n \in \mathbb{N}_0).$$

Define $A_n: D_0 \to \ell^1$ analogously to $A_\infty$ with $\alpha_{[n]}_{jk}$ replacing $\alpha_{jk}$, for $n \in \mathbb{N}_0$. Note that in this case the operators $B_n := A_n - A_0$ are bounded (positive) operators.
4.3 Theorem. The operators $A_n$ generate substochastic $C_0$-semigroups $S_n$ on $\ell^1$, and $\|S_n(t)x - S(t)x\| \to 0$ as $n \to \infty$ for each $x \in \ell^1$. These semigroups leave $\ell^{11}$ invariant and induce $C_0$-semigroups on $\ell^{11}$, and $\|S_n(t)x - S(t)x\|_{11} \to 0$ as $n \to \infty$ for each $x \in \ell^{11}$. The convergence is monotone increasing, if $x$ is positive.

In order to illustrate Assumption 4.1, we consider a birth and death process with catastrophes and immigration.

We assume that $(\alpha_{jk})$ is a Kolmogorov matrix satisfying
\[
\alpha_{jk} = 0 \quad \text{for all} \quad j, k \in \mathbb{N}_0 \text{ with } j > k + 1.
\] (4.2)

This means that birth rates are such that populations can only increase by one. On the other hand the model allows drastic decreases, including catastrophes wiping out the whole population. This model was also treated in [17; § 4, Example 4]. We indicate a condition implying Assumption 4.1.

4.4 Lemma. If (4.2) is satisfied then Assumption 4.1 holds if there exists $a > 1$ such that
\[
s := \sup_{k \geq 0} \frac{1}{k + 1} \left( a \alpha_{k+1,k} - \sum_{j=0}^{k-1} (k - j)\alpha_{jk} \right) < \infty
\] (4.3)

Proof. The matrix $(\alpha_{jk})$ being a Kolmogorov matrix satisfying (4.2) implies
\[
\sum_{j=0}^{\infty} j\alpha_{jk} = \alpha_{k+1,k} + \sum_{j=0}^{k-1} (j - k)\alpha_{jk} \quad (k \in \mathbb{N}_0).
\]

Because of this equation Assumption 4.1 can be reformulated as
\[
(1 + \varepsilon)\alpha_{k+1,k} \leq c(1 + k) + \sum_{j=0}^{k-1} (k - j - \varepsilon)\alpha_{jk} \quad (k \in \mathbb{N}_0).
\]

It is not difficult to see that (4.3) implies these inequalities if $(1 + \varepsilon)a \leq (1 - \varepsilon)$, i.e. $0 < \varepsilon \leq \frac{a-1}{a+1}$, and $c \geq \frac{1-a}{a} s$. \qed

5 A measure-valued generalization of Kolmogorov’s differential equations

Let $(\Omega, A)$ be a measurable space. We consider a measure-valued generalization of Kolmogorov’s differential equations,
\[
\frac{d}{dt} \mu(t)(\Gamma) = \int_{\Omega} K(\Gamma, x)\mu(t)(dx) - \int_{\Gamma} K(\Omega, x)\mu(t)(dx) \quad (\Gamma \in A).
\]

$K$ is a transition measure kernel: For each $x \in \Omega$, $K(\cdot, x)$ is a non-negative finite measure on $(\Omega, A)$ and for each $\Gamma \in A$, $K(\Gamma, \cdot)$ is $A$-measurable on $\Omega$. The solution $\mu(t)$ takes its values in the Banach space $X$ of signed measures on $(\Omega, A)$ which have bounded variation. If
the values are probability measures, they are associated with the transition probabilities of a Markov jump process ([14; X.3], [12; 4.2]).

Let \( h := K(\Omega, \cdot) \). Then the operator \( A_0 \) given by

\[
D(A_0) := \{ \mu \in X; \int_{\Omega} h(x)|\mu|(dx) < \infty \},
\]

\[
(A_0\mu)(\Gamma) := -\int_{\Gamma} h(x)\mu(dx),
\]

generates the \( C_0 \)-semigroup \( S_0 \),

\[
(S_0(t)\mu)(\Gamma) = \int_{\Gamma} e^{-th(x)}\mu(dx).
\]

The denseness of \( D(A_0) \) can be seen as follows: Set \( \Omega_n = \{ x \in \Omega; h(x) \leq n \} \). Let \( \mu \in X \). Define \( \mu_n(\Gamma) = \mu(\Gamma \cap \Omega_n) \). Then \( \mu_n \in D(A_0) \) and \( \mu_n \to \mu \) as \( n \to \infty \) because \( \Omega \) is the union of the increasing sequence of sets \( \Omega_n \).

We define a positive linear operator \( B : D(A_0) \to X \),

\[
(B\mu)(\Gamma) = \int_{\Omega} K(\Gamma, x)\mu(dx).
\]

Let \( \eta : \Omega \to [0, \infty) \) be \( A \)-measurable. If we choose \( X_1 \) to be the space of signed measures \( \mu \) with \( \| \mu \|_1 = \int_{\Omega}(1 + \eta(x))|\mu|(dx) < \infty \), we obtain the following result from Theorem 2.2, Proposition 2.6 and Theorem 2.7. Notice that the density of \( X_1 \) in \( X \) follows in the same way as the density of \( D(A_0) \).

5.1 Proposition. (a) There exists a smallest positive \( C_0 \)-semigroup \( S \) on \( X \) whose generator extends \( A_0 + B \). \( S \) is substochastic.

(b) Assume that there exist positive constants \( c, \varepsilon \) such that

\[
\int_{\Omega} (\eta(y) - \eta(x))K(dy, x) \leq c(1 + \eta(x)) - \varepsilon K(\Omega, x) \quad \text{for all } x \in \Omega.
\]

Then \( S \) is a stochastic \( C_0 \)-semigroup which is generated by the closure of \( A_0 + B \), leaves \( X_1 \) invariant and induces a \( C_0 \)-semigroup on \( (X_1, \| \cdot \|_1) \).

We now show that the semigroup \( S \) is associated with a Markov transition function. We first note that \( Y = \text{BM}(\Omega) \), the space of bounded \( A \)-measurable functions on \( \Omega \) with the supremum norm, can be identified with a subspace of \( X^* \). The dominated convergence theorem implies that \( \text{BM}(\Omega) \) is weakly* sequentially closed. \( \text{BM}(\Omega) \) is invariant under \( ((\lambda - A_0)^{-1})^* \) and \( (B(\lambda - A_0)^{-1})^* \), because

\[
((\lambda - A_0)^{-1})^* f(x) = \frac{f(x)}{\lambda + h(x)},
\]

\[
(B(\lambda - A_0)^{-1})^* f(x) = \int_{\Omega} \frac{f(y)}{\lambda + h(x)}K(dy, x)
\]

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for all \( x \in \Omega, f \in \text{BM}(\Omega) \). By Proposition 2.4, \( \text{BM}(\Omega) \) is invariant under \( S^* \). Set

\[
P_t(\Gamma, x) := (S(t)\delta_x)(\Gamma) \quad (\Gamma \in \mathcal{A}),
\]

where \( \delta_x \) is the Dirac measure concentrated at \( x \). Then \( P_t(\cdot, x) \) is a non-negative measure on \( \mathcal{A} \) with values in \([0, 1]\). Since also \( P_t(\Gamma, \cdot) = S^*(t)\chi_{\Gamma} \) and \( \text{BM}(\Omega) \) is invariant under \( S^*(t) \), \( P_t(\Gamma, \cdot) \in \text{BM}(\Omega) \) and

\[
(S(t)\mu)(\Gamma) = \int_{\Omega} P_t(\Gamma, x)\mu(dx).
\]

The semigroup property of \( S \) implies that \( P \) satisfies the Chapman-Kolmogorov equations

\[
P_{t+s}(\Gamma, x) = \int_{\Omega} P_t(\Gamma, y)P_s(dy, x),
\]

i.e., \( P \) is a Markov transition function [22; Sec. 3.2]. Since \( S \) is a \( C_0 \)-semigroup, the definition (5.1) shows that for fixed \( x \in \Omega \), the function \( t \mapsto P_t(\Gamma, x) \) is continuous on \([0, \infty)\), uniformly for \( \Gamma \in \mathcal{A} \). If the assumption in Proposition 5.1(b) is satisfied, then \( P_t(\cdot, x) \) is a probability measure, and therefore \( P \) is a normal Markov transition function.

**Appendix: A version of the Miyadera perturbation theorem**

In this appendix we assume that \( X \) is an ordered Banach space with a generating (closed) cone \( X_+ \). We start with an observation that will be needed in the proof of the main result of this section.

**A.1 Remarks.** (a) Under the above hypothesis the positive cone \( X_+ \) is non-flat, i.e., there exists \( M \geq 1 \) such that for all \( x \in X \), \( \|x\| \leq 1 \), there exist \( x_{\pm} \in X_+ \) with \( \|x_{\pm}\| \leq M \), \( x = x_+ - x_- \); cf. [9; Proposition 19.1(d)], [10; p. 265].

This immediately implies that any linear operator \( C: X \to X \) satisfying \( c := \sup\{\|Cx\|; x \in X_+, \|x\| \leq 1\} < \infty \) is bounded, \( \|C\| \leq 2Mc \).

(b) Let \( A \) be the generator of a positive \( C_0 \)-semigroup \( S \) on \( X \). Then (a) can be strengthened as follows.

Let \( C: D(A) \to X \) be linear, \( A \)-bounded, \( c := \sup\{\|Cx\|; x \in D(A)_+, \|x\| \leq 1\} < \infty \). Then \( C \) uniquely extends to an operator \( C \in L(X) \) satisfying \( \|C\| \leq 2Mc, \sup\{\|Cx\|; x \in X_+, \|x\| \leq 1\} = c \).

In fact, let \( x \in D(A), \|x\| \leq 1 \). There exist \( x_{\pm} \in X_+, \|x_{\pm}\| \leq M \), \( x = x_+ - x_- \). For \( \lambda \) larger than the type of \( S \) one has

\[
\lambda(\lambda - A)^{-1}x = \lambda(\lambda - A)^{-1}x_+ - \lambda(\lambda - A)^{-1}x_-,
\]

\[
\lambda(\lambda - A)^{-1}x_{\pm} \in D(A)_+.
\]

Also, \( \lambda(\lambda - A)^{-1}x_{\pm} \to x_{\pm} (\lambda \to \infty) \), and \( \lambda(\lambda - A)^{-1}x \to x \) in the \( A \)-graph norm \( (\lambda \to \infty) \). Taking \( \lambda \to \infty \) in

\[
\|C\lambda(\lambda - A)^{-1}x\| \leq c(\|\lambda(\lambda - A)^{-1}x_+\| + \|\lambda(\lambda - A)^{-1}x_-\|)
\]

we obtain \( \|Cx\| \leq c(\|x_+\| + \|x_-\|) \leq 2Mc \).

Now \( D(A) \) being dense implies that \( C \) extends as asserted. Since \( D(A)_+ \) is dense in \( X_+ \) one also obtains the last equality.
A.2 Theorem. Let $S_0$ be a positive $C_0$-semigroup on $X$, with generator $A_0$. Let $B: D(A_0) \to X$ be positive. Assume that there are constants $0 < \alpha \leq \infty$, $\gamma \in [0, 1)$ such that
\[
\int_0^\alpha \|BS_0(t)x\| \, dt \leq \gamma \|x\| \quad (x \in D(A_0)).
\]  
(A.1)

Then $A_0 + B$ is the generator of a positive $C_0$-semigroup $S$.

A.3 Remarks. (a) If $X, Y$ are ordered Banach spaces such that $X_+$ is generating (i.e., $X = X_+ - X_+$) and $Y_+$ is proper (i.e., $Y_+ \cap (-Y_+) = \{0\}$), then any positive linear operator $T: X \to Y$ is bounded; cf. [1; Appendix], [18; Theorem 2.1].

This implies that, in Theorem A.2, the operator $B$ is $A_0$-bounded.

(b) Note that (A.1) only implies
\[
\int_0^\alpha \|BS_0(t)x\| \, dt \leq 2M\gamma \|x\| \quad (x \in D(A_0))
\]
(with a proof as in Remark A.1(b)), so the assertion of Theorem A.2 is not a direct consequence of the Miyadera perturbation theorem; cf. [23; Theorem 1], [11; chap. 3, Theorem 3.14]. (For the application of the Miyadera perturbation theorem the constant $2M\gamma$ would have to be $< 1$.)

(c) The proof will show that
\[
\sup \{\|S(t)x\|; 0 \leq t < \alpha, x \in X_+, \|x\| \leq 1\} \leq \frac{1}{1-\gamma} \sup \{\|S_0(t)x\|; 0 \leq t < \alpha, x \in X_+, \|x\| \leq 1\}.
\]

Proof of Theorem A.2. As mentioned above in Remark A.3(a), the operator $B$ is $A_0$-bounded.

By induction we define strongly continuous mappings $S_n: [0, \alpha) \to L(X)$ satisfying
\[
S_n(t)x := \int_0^t S_{n-1}(t-s)BS_0(s)x \, ds \quad (x \in D(A_0), 0 \leq t < \alpha),
\]  
(A.2)

\[
\|S_n(t)x\| \leq \gamma^n \|x\| \quad (x \in X_+, 0 \leq t < \alpha),
\]

for all $n \in \mathbb{N}$. Indeed, the linear mappings $S_n(t)$ defined by (A.2) belong to $L(D(A_0), X)$, where $D_{A_0}$ denotes $D(A_0)$ provided with the graph norm. The induction hypothesis implies
\[
\|S_n(t)x\| \leq \gamma^{n-1} \int_0^t \|BS_0(s)x\| \, ds \leq \gamma^n \|x\| \quad (x \in D(A_0)_+).
\]

Then use Remark A.1(b).

The series $S(t) := \sum_{n=0}^\infty S_n(t)$ is norm convergent, uniformly for $0 \leq t < \alpha$, and as a consequence, $S$ is strongly continuous. The proof that $S$ satisfies the semigroup property on $[0, \alpha)$ and that $S$ can be extended to a $C_0$-semigroup is the same as in [23; proof of Theorem 1].

It is easy to see that the generator $A$ of $S$ is an extension of $A_0 + B$ (cf. [23; Lemma 3]). Therefore
\[
(\lambda - A_0)^{-1} = (\lambda - A)^{-1}(I - B(\lambda - A_0)^{-1})
\]
for large $\lambda$. In order to show $D(A) = D(A_0)$ (then $A = A_0 + B$) it therefore is sufficient to show $\text{spr}(B(\lambda - A_0)^{-1}) < 1$ (spr denoting the spectral radius). Let $\gamma' \in (\gamma, 1)$. The estimate
as on [23; p. 168] shows that there exists $\lambda > 0$ such that $\| B(\lambda - A_0)^{-1} x \| \leq \gamma' \| x \|$ for all $x \in D(A_0)$. Since $B(\lambda - A_0)^{-1}$ is bounded we obtain $\| B(\lambda - A_0)^{-1} x \| \leq \gamma' \| x \|$ for all $x \in X_+$. Therefore $\|(B(\lambda - A_0)^{-1})^n x\| \leq \gamma'^n \| x \|$ for all $x \in X_+, n \in \mathbb{N}$ (it is at this point where the positivity of $B$ is used), and finally $\|(B(\lambda - A_0)^{-1})^n \| \leq 2M \gamma'^n$ ($n \in \mathbb{N}$). These inequalities show $\text{spr}(B(\lambda - A_0)^{-1}) \leq \gamma'$.

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