On the sum of two closed subspaces

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Abstract
If $U, V$ are closed subspaces of a Fréchet space, then $E$ is the direct sum of $U$ and $V$ if and only if $E'$ is the algebraic direct sum of the annihilators $U^\circ$ and $V^\circ$. We provide a simple proof of this (possibly well-known) result.

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Introduction
The starting point of the present note was a statement in [2; second paragraph of the proof of Lemma 5]: If $E$ is a Banach space, and $E'$ is the algebraic direct sum of two $\sigma(E', E)$-closed subspaces $M$ and $N$ of $E'$, then $E'$ is the topological direct sum of $M$ and $N$ with respect to the topology $\sigma(E', E)$. This is stated without a reference, but after some research a reference was found in Luxemburg [1]. Unfortunately, in this reference there is a gap in a crucial argument, and it was not immediately clear how to mend this gap. We refer to Remark 1.6(a) for more explicit explanations.

The objective of this note is to give a short proof of the statement mentioned initially and an extension stated in [1; Theorem (2.4)].

1 The sum of closed subspaces
The following is the main result of this note.

1.1 Theorem. Let $E$ be a Fréchet space, and let $U, V$ be closed subspaces of $E$. Then the following statements are equivalent.

(a) $E$ is the algebraic (or equivalently, the topological) direct sum of $U$ and $V$,
(b) $E'$ is the algebraic direct sum of $U^\circ$ and $V^\circ$.

If one of these conditions is verified, then $E'$ is the topological direct sum of $U^\circ$ and $V^\circ$ with respect to the topology $\sigma(E', E)$.
**Proof.** (a) ⇒ (b). We quote [3; III, 2.1, Corollary 3] for the fact that the hypotheses imply that E is the topological direct sum of U and V.

Let \( P_U \) denote the projection of E onto U along V. Then the transpose \( P_U' \) of \( P_U \) is the projection of \( E' \) onto \( V^\circ \) along \( U^\circ \), and is continuous with respect to the topology \( \sigma(E', E) \) (cf. [3; IV, 7.4]). This shows that \( E' \) is the topological direct sum of \( U^\circ \) and \( V^\circ \) with respect to the topology \( \sigma(E', E) \).

(b) ⇒ (a). The hypothesis \( E' = U^\circ + V^\circ \) implies that \( U \cap V = \{0\} \). Also, \( U^\circ \cap V^\circ = \{0\} \) implies that \( U + V \) is dense in E (by a standard application of the Hahn-Banach theorem), and therefore the dual of \( U + V \) with respect to the metric topology inherited from E is equal to \( E' \). In the dual pair \( \langle U + V, E' = U^\circ + V^\circ \rangle \), it is easy to show that the projections \( P_U : U + V \rightarrow U \) along V and \( P_{V^\circ} : E' \rightarrow V^\circ \) along \( U^\circ \) are transposes of each other. Applying [3; IV, 7.4], one obtains that \( P_U \) is continuous with respect to the Mackey topology \( \mu(U + V, E') \). However, the circumstance that \( U + V \) is a metric locally convex space implies that \( \mu(U + V, E') \) is just the metric topology inherited from E (cf. [3; IV, 3.4]). This means that \( P_U \) is continuous with respect to the Fréchet space topology. It is easy to see that this implies that \( U + V \) is closed in E, and therefore \( U + V = E \). \( \square \)

**1.2 Remarks.** (a) We refer to [3; IV, 2.1, Example] for a discussion relating the properties of being the algebraic or topological direct sum with respect to the weak topology, for subspaces of \( F \), in a dual pair \( \langle F, G \rangle \). In fact, in the proof of ‘(a) ⇒ (b)’ given above, this reference could have been used, after noting that the projection \( P_U \) is \( \sigma(E, E') \)-continuous.

(b) It might be worth mentioning that in the case that \( E \) is a Banach space a simpler proof for part of the implication ‘(b) ⇒ (a)’ is available. In this case one can argue that \( E' \) is the topological direct sum of \( U^\circ \) and \( V^\circ \) with respect to the norm topology of \( E' \) (because \( U^\circ \) and \( V^\circ \) are closed in \( E' \)). Therefore the projection \( P_{V^\circ} \) from \( E' \) onto \( V^\circ \)along \( U^\circ \) is continuous, and this implies that its transpose \( P_{V^\circ}' \) is continuous in \( E'' \). Now, an easy computation shows that the restriction of \( P_{V^\circ}' \) to \( U + V \) is just the projection \( P_U \) of \( U + V \) onto \( U \) along \( V \).

The statement which was mentioned in the Introduction is now an immediate consequence of the previous theorem, as follows.

**1.3 Corollary.** Let \( E \) be a Fréchet space, and let \( E' \) be the algebraic direct sum of two \( \sigma(E', E) \)-closed subspaces \( M \) and \( N \) of \( E' \). Then \( E' \) is the topological direct sum of \( M \) and \( N \) with respect to the topology \( \sigma(E', E) \).

**Proof.** With \( U := M^\circ \) and \( V := N^\circ \), the bipolar theorem implies that \( U^\circ = M^{\circ \circ} = M \) and \( V^\circ = N^{\circ \circ} = N \). Then Theorem 1.1 yields the assertion. \( \square \)

The following corollary is the statement of [1; Theorem (2.4)] mentioned in the Introduction.

**1.4 Corollary.** Let \( E \) be a Fréchet space, and let \( U, V \subseteq E \) be closed subspaces. Then the sum \( U + V \) is closed in \( E \) if and only if \( U^\circ + V^\circ \) is \( \sigma(E', E) \)-closed in \( E' \).
The corollary will be proved below after two further preparations.

In [1; Theorem (2.4)], the second of the equivalent conditions in Corollary 1.4 was formulated as ‘\((U \cap V)^\circ = U^\circ + V^\circ\)’. We add the explanation why these formulations are equivalent. Indeed, if \(\langle E, F \rangle\) is a dual pair and \(U, V \subseteq E\) are \(\sigma(E, F)\)-closed subspaces, then

\[
(U \cap V)^\circ = U^\circ + V^\circ, \quad (U \cap V)^\circ = U^\circ + V^\circ^{\sigma(F,E)}. \tag{1.1}
\]

This is mentioned in [3; IV, 1.5, remarks after Corollary 4],

We will also need the following (certainly well-known) fact from quotient spaces.

1.5 Lemma. Let \(E\) be a topological vector space, let \(U \subseteq E\) be a closed subspace, \(L \subseteq U\) a subspace, and let \(Q: E \to E/L\) be the quotient map. Then \(Q(U)\) is a closed subspace of \(E/L\).

Proof. The hypothesis \(L \subseteq U\) implies that \((E/L) \setminus Q(U) = Q(E \setminus U)\), and as the quotient map is open, one obtains the assertion.

Proof of Corollary 1.4. (i) Assume first that \(U + V\) is dense in \(E\). Let \(Q: E \to E/(U \cap V) =: \hat{E}\) be the quotient map. Note that \(\hat{E}' = (U \cap V)^\circ\), and that \(Q': \hat{E}' \to E'\) is the injection. It is easy to check that \(Q(U)^\circ = U^\circ\) (and \(Q(V)^\circ = V^\circ\)). Lemma 1.5 implies that \(Q(U)\) and \(Q(V)\) are closed subspaces of \(\hat{E}\). We have the following equivalences: \(U + V = \hat{E} \iff Q(U) + Q(V) = Q(U + V) = \hat{E} \iff U^\circ + V^\circ = Q(U)^\circ + Q(V)^\circ = \hat{E}' \iff U^\circ + V^\circ\ \sigma(E', E)\)-closed in \(E'\), where we have applied Theorem 1.1 in the middle equivalence and (1.1) in the third.

(ii) Now we treat the general case. Let \(\hat{E} := \frac{U + V}{U\cap V}\

and \(\sigma(\hat{E}', \hat{E})\) is the quotient topology of \((E', \sigma(E', E))\). Step (i) implies that \(U + V\) is closed (i.e., that \(U + V = \hat{E}\) if and only if \(U^\circ + V^\circ\) is \(\sigma(\hat{E}', \hat{E})\)-closed (with polars taken in the dual pair \((E', (U + V)^\circ)\), and \(U, \hat{V}\) denoting \(U, V\) as subsets of \(E\)). Let \(J: \hat{E} \to E\) denote the embedding; then \(J': E' \to \hat{E}' = E'/((U + V)^\circ)\) is the quotient map. What we still have to show is that \(U^\circ + \hat{V}^\circ\) is \(\sigma(\hat{E}', \hat{E})\)-closed if and only if \(U^\circ + V^\circ = J(U)^\circ + J(\hat{V})^\circ\) is \(\sigma(E', E)\)-closed. Now, it is not difficult to show that \(J(U)^\circ = J^{-1}(\hat{U}^\circ)\) (and \(J(\hat{V})^\circ = J^{-1}(\hat{V}^\circ)\)), and this implies that

\[
U^\circ + V^\circ = J^{-1}(\hat{U}^\circ + \hat{V}^\circ), \quad J'(U^\circ + V^\circ) = U^\circ + \hat{V}^\circ.
\]

Now Lemma 1.5 implies that \(U^\circ + V^\circ\) is \(\sigma(E', E)\)-closed if and only if \(U^\circ + \hat{V}^\circ\) is \(\sigma(\hat{E}', \hat{E})\)-closed, and this concludes the proof.

1.6 Remarks. (a) The author could not follow the reasoning in the second part of the proof of [1; Theorem (2.4)]. In fact there are two steps where it is not clear how to interpret the reasoning: The first is that in [1] it is claimed that ‘the mappings \(x + y \mapsto x\) ... of \((U + V)/(U \cap V)\) onto \(U\) ... are continuous’. The example \(U = V\) immediately shows that such mappings do not exist. The
second is that the continuity of the mappings (if interpreted suitably; see below) would be with respect to the weak topology, in view of the previous explanations in [1]. However, in order to continue the argument, one would need continuity with respect to the metric topology.

The author is indebted to W. Ruess for communicating the following correcting versions of the missing arguments. Concerning the first point, the addition \( a: U \times V \to U + V \) gives rise to a bijective map \( a_0: (U \times V)/\ker a \to U + V \). Then the fact that \( a \) is open with respect to the weak topologies ([1; Theorem 2.3]) means that \( a_0^{-1}: U + V \to (U \times V)/\ker a \) is continuous with respect to the weak topologies. Having established this property, the second point can be treated similarly to part of the proof of Theorem 1.1'\( (b) \Rightarrow (a)' \), as follows. One concludes that \( a_0^{-1} \) is also continuous with respect to the Mackey topologies and uses that the Mackey topologies are the metric topologies. Now, assuming that \( u_n + v_n \to u \in E \), then \( \{(u_n, v_n)\} \) is a Cauchy sequence in the quotient space, hence convergent to some \( [(u, v)] \), and therefore \( x = u + v \in U + V \).

It might be worth mentioning that Theorem 1.1 can be deduced from [1; Theorem (2.4)], as reformulated in Corollary 1.4. Indeed, either of the conditions \( (a) \) or \( (b) \) of Theorem 1.1 implies that both \( U + V \) and \( U^\circ + V^\circ \) are algebraic direct sums, \( U + V \) dense in \( E \), and \( U^\circ + V^\circ \) dense in \( E' \) with respect to \( \sigma(E', E) \). Then Corollary 1.4 shows that \( (a) \) and \( (b) \) are equivalent, and [3; IV, 2.1, Example] implies that the sum \( U^\circ + V^\circ = E' \) is topologically direct with respect to \( \sigma(E', E) \).

(b) Even though the arguments in [1] now have been clarified, the author thinks that it is worth while to present a somewhat different approach to the problem. In particular, it may be of interest to have a more direct proof of the statement in Theorem 1.1 and then to derive Corollary 1.4 as a consequence.

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References

