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On positive invertibility of operators and their decompositions

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In Banach spaces ordered by a normal cone that contains interior points the positive invertibility of operators is studied. If there exists a uniformly positive functional then any positively invertible operator \( A \) possesses a \( B \)-decomposition, i.e., a positive decomposition \( A = U - V \) with the properties: \( U^{-1} \) exists, \( \text{tr}(U^{-1}) \geq 0 \), \( Ax \geq 0 \), \( Ux \geq 0 \) imply \( x \geq 0 \) and \( \text{tr}(Vu^{-1}) < 1 \). Earlier it was shown that the existence of a \( B \)-decomposition with \( \text{tr}(Vu^{-1}) < 1 \) is sufficient for the positive invertibility of the operator \( A \). Peris’ result is obtained as a special case of the main theorem. The decomposition is demonstrated for a positively invertible operator in a Banach space ordered by an ice cream cone.

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1 Introduction

Let \((X, X_+, \| - \|)\) and \((Y, Y_+, \| - \|)\) be ordered normed spaces. Denote by \( L(X,Y) \) the vector space of all linear continuous operators \( A : X \rightarrow Y \). We shall write \( L(X) \) for \( L(X, X) \). An operator \( A \in L(X, Y) \) is said to be positive \((A \geq 0)\) if \( A(X_+) \subset Y_+ \).

Inverse-monotone operators (i.e., \( Ax \geq 0 \) implies \( x \geq 0 \)), and in particular matrices of such kind, are of interest in connection with the existence of a positive solution for equations

\[
Ax = y, \quad \text{where} \quad y \in Y_+
\]

(see for e.g. [8], [9], [15], [16], [23]). M. I. Gil studied the case of integral operators in [11] and the positive invertibility of some operators in separable Hilbert lattices in [13]. Concerning matrices much effort has been made in order to obtain necessary and/or sufficient conditions for the inverse monotony (see [6], [10], [12], [17], [21] and others). The paper [15] contains interesting results of majorizing and minorizing types. More exactly, in order to state at least one of these results, the positive invertibility of an operator \( C \) (this means \( C^{-1} \) exists in \( L(X) \) and \( C^{-1} \geq 0 \)) in some Banach space ordered by a special cone is guaranteed provided \( A \leq C \leq B \) and the operators \( A, B \) are positively invertible. Based on a result of J. Peris ([21]) in a former paper by the author ([27]) there was proved that a certain spectral property which should share all positive decompositions of an invertible operator acting in an appropriate ordered Banach space is a sufficient (and under some additional requirement on the operator also a necessary) condition for the operator to be positively invertible. This idea was used by T. Kurmayya and K. C. Sivakumar in [18] to establish the positivity of the Moore–Penrose inverses ([7], [14]) for operators in Hilbert lattices.

Another kind of decomposition of an operator, quite popular in the theory of positive matrices, is used by J. Peris and B. Subiza in [22] for the case of weak monotonicity of matrices (i.e., for all \( x \in \mathbb{R}^n \) such that \( Mx \geq 0 \) there exists \( y \in \mathbb{R}^n_+ \) satisfying \( My = Mx \)).

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2 Preliminaries

Let \((X, X_+, \|\cdot\|)\) be an ordered normed space, where throughout the cone \(X_+\) is understood to be closed. Briefly we will write \(X\) instead of \((X, X_+, \|\cdot\|)\). In the vector space \(L(X, Y)\) of all linear continuous operators \(A: X \to Y\), where \(Y\) stands for the ordered normed space \((Y, Y_+, \|\cdot\|)\), the subset \(L_+(X, Y) = \{ A \in L(X, Y) : A(X_+) \subseteq Y_+ \}\) is, in general, a closed wedge and, under the assumption that the linear set \(X_+ - X_-\) is dense in \(X\), it is a cone in \(L(X, Y)\) (see [24]). If \(X = Y\) (this means also \(Y_+ = X_+\) and the identity of the norms) then \(L(X, X)\) is denoted by \(L(X)\) and \(L_+(X, Y)\) by \(L_+(X)\).

If \(Y = \mathbb{R}^1\) and \(Y_+ = \mathbb{R}^1_+\), then \(L_+(X, Y)\) is denoted by \(X_+\) and is called the dual wedge or cone, respectively.

Moreover, if the cones \(X_+\) and \(Y_+\) are both normal, \(\text{int}(Y_+) \neq \emptyset\) and \((Y, Y_+)\) is a Dedekind complete vector lattice then the wedge \(L_+(X, Y)\) is reproducing (or generating) ([3]–[5], [25]), i.e., any \(A \in L(X, Y)\) can be represented as the difference of two positive operators: \(A = U - V\), where \(U, V \in L_+(X, Y)\).

If \(A \in L(X, Y)\) is invertible then the continuity of \(A^{-1}\) follows either from Banach’s theorem (if \(X\) and \(Y\) are Banach spaces) or, if \(A^{-1} \geq 0\) is known, from continuity theorems for positive operators\(^\dagger\) (see [25], [1]).

For an operator \(A \in L(X)\) the spectrum is denoted by \(\sigma(A)\) and the spectral radius is defined as \(r(A) = \max \{|\lambda| : \lambda \in \sigma(A)\}\). The spectral radius of an invertible operator \(A\) is positive. Indeed, \(1 = r(I) \leq r(A)r(A^{-1})\) implies \(r(A) > 0\) and \(r(A^{-1}) > 0\). If an operator \(A\) is positively invertible then \(r(A^{-1}) \in \sigma(A^{-1})\) (see [17, Thm. 8.1]).

The next theorem provides some well-known conditions on an operator \(C\) in a Banach space \(X\) in order to guarantee the invertibility of the operator \(I - C\) in \(L(X)\).

**Theorem 2.1** Let \(X\) be a Banach space and \(C: X \to X\) be a continuous linear operator on \(X\).

Consider the following properties:

- a) the spectral radius of \(C\) satisfies \(r(C) < 1\);
- b) \(C\) is quasinilpotent, i.e., \(\lim_{n \to \infty}\|C^n\| = 0\);
- c) there exists the inverse operator \((I - C)^{-1}\).

Then the following implications hold: (a) \(\implies\) (b) \(\implies\) (c).

**Proof.** a) \(\implies\) b). By the Gelfand formula and the assumption one has \(r(C) = \lim_{n \to \infty} \sqrt[n]{\|C^n\|} < 1\). For a fixed real number \(q\) such that \(r(C) < q < 1\) there exists a number \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) one has \(\sqrt[n]{\|C^n\|} < q\) and consequently \(\|C^n\| < q^n\). From \(q < 1\) there follows \(\|C^n\| \to 0\).

b) \(\implies\) c). For each \(n\) one has \((I - C)(I + C + C^2 + \cdots + C^n) = I - C^{n+1}\). If \(C^n \to 0\) then \((I - C)\sum_{n=0}^{\infty}C^n = I\) and

\[(I - C)^{-1} = I + C + C^2 + \cdots + C^n + \ldots\] (2.1)

This proves the theorem.

**Corollary 2.2** Condition b) implies also the existence of the operator \((I + C)^{-1}\), where

\[(I + C)^{-1} = I - C + C^2 + \cdots + (-1)^nC^n + \ldots\] (2.2)

If \(X = (X, X_+, \|\cdot\|)\) is an ordered Banach space and the operator \(C\) is positive, i.e., \(C \in L_+(X)\), then the situation improves. Namely, it is easily seen from (2.1), that a) implies even \((I - C)^{-1} \geq 0\). If the operator \(C\) is nilpotent, i.e., \(C^k = 0\) for some \(k \in \mathbb{N}\), then \(I + C\) is invertible, where \((I + C)^{-1} = I - C + C^2 + \cdots + (-1)^{k-1}C^{k-1}\). Formula (2.2) shows that \((I + C)^{-1}\) is also positive, if the operator \(-C \geq 0\). Indeed, in this case \(C^{2n} \geq 0\) and \(-C^{2n-1} \geq 0\) for any \(n \in \mathbb{N}\). The implication c) \(\implies\) a) holds if the cone in the ordered Banach space has additional properties. Moreover, if the operator \(C\) is compact and satisfies \(r(C) > 0\) then the number \(r(C)\) turns out to be an eigenvalue of \(C\) possessing a positive eigenvector\(^\dagger\). More precisely one has (see [17, Thm. 9.2 and § 25]).

\(^\dagger\) If \(X\) is an ordered Banach space such that each positive linear functional on \(X\) is continuous and \(Y\) is an ordered Banach space with a closed cone then any linear positive operator \(A: X \to Y\) is continuous (Lozanovski’s Theorem). The condition for \(X\) holds e.g. if \(\text{int}(X_+) \neq \emptyset\) or if \(X_+\) is closed and reproducing. The norm-completeness of \(Y\) can be removed if the cone \(Y_+\) in the ordered normed space \(Y\) is assumed to be normal. Of course, in case of normed lattices the situation is much simpler: If \(X\) is a Banach lattice and \(Y\) any normed lattice then each positive operator is continuous.

\(^\dagger\) Such eigenvalues are called Perron–Frobenius-eigenvalues.

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Theorem 2.3 Let $(X,X_+,[\|\cdot\|])$ be an ordered Banach space and $C : X \to X$ a continuous linear operator on $X$ such that $C \geq 0$.

(i) Then $r(C) < 1$ implies $(I - C)^{-1} \geq 0$ and, if the cone $X_+$ is normal and reproducing, also vice versa, i.e., the existence of $(I - C)^{-1}$ and $(I - C)^{-1} \geq 0$ imply $r(C) < 1$.

(ii) Let the cone $X_+$ satisfy the condition $X_+ - X_+ \subset X$. If $C$ is compact, $r(C) > 0$ and $(I - C)^{-1} \geq 0$ then $r(C)$ is an eigenvalue of $C$ possessing a positive eigenvector and $r(C) < 1$.

Proof. (i) If $r(C) < 1$ and $C \geq 0$ then by Theorem 2.1 the operator $(I - C)^{-1}$ exists and $(I - C)^{-1} \geq 0$ immediately follows from (2.1). The inverse statement is [17, Thm. 25.1].

(ii). The compactness of the positive operator $C$ implies the existence of a positive eigenvector, so there exists a nonzero vector $x_0 \in X_+$ such that $Cx_0 = r(C)x_0$. If there would be $r(C) = 1$ then $(I - C)x_0 = 0$ contradicts the invertibility of $I - C$. Consequently, $r(C) \neq 1$ and $(I - C)x_0 = (1 - r(C))x_0$ implies $\frac{1}{1 - r(C)}x_0 = (I - C)^{-1}x_0$. Since $x_0 > 0$ and $(I - C)^{-1} \geq 0$ there must hold $r(C) < 1$.

Corollary 2.4 The operator $sl - C$ $(s > 0)$ is positively invertible if and only if $r(C) < s$.

Theorem 2.5 ([16, Thm. 1], [17, Thm. 25.4]). Let $X = (X,X_+,[\|\cdot\|])$ be an ordered Banach space with a normal cone $X_+$ that satisfies $\text{int}(X_+) \neq \emptyset$. Let $C,B : X \to X$ be two linear continuous operators, where $C \leq B$ and $B$ is positively invertible. Then $C$ is positively invertible if and only if $C(X_+) \cap \text{int}(X_+) \neq \emptyset$.

3 Decompositions and positive invertibility

In what follows $X = (X,X_+,[\|\cdot\|])$ will be an ordered normed space and $A \in L(X)$.

Definition 3.1 A decomposition $A = U - V$ of $A$ is said to be positive, if $U \geq 0$, $V \geq 0$ (in this case the operator $A$ is regular), positive regular, if it is positive, $U^{-1}$ exists and $U^{-1} \geq 0$.

$B$-decomposition$^3$, if it is positive and satisfies the conditions

a) there exists $U^{-1}$;

b) $VU^{-1} \geq 0$;

c) $Ax \geq 0$, $Ux \geq 0$ imply $x \geq 0$.

The notion of a $B$-decomposition is a generalization (to the infinite-dimensional case) of a $B$-splitting introduced in [21] for square matrices.

If $A$ is a positively invertible matrix then there might exist positive decompositions $A = U - V$ with the following properties:

a) there exists $U^{-1}$ and $V$ is not invertible or there exists $V^{-1}$ and $U$ is not invertible, even if $A \geq 0$ and $A^{-1} \geq 0$,

b) both $U^{-1}$, $V^{-1}$ exist,

c) $U$, $V$ are both not invertible.

Examples 3.2

a)

$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, \hspace{1cm} $A^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, \hspace{1cm} $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, \hspace{1cm} $V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$,

$A = \begin{pmatrix} 3 & -2 & 1 \\ -3 & 2 & 0 \\ 0 & 3 & -4 \end{pmatrix}$, \hspace{1cm} $A^{-1} = \begin{pmatrix} 8 & 5 & 2 \\ 9 & 9 & 9 \\ 4 & 4 & 1 \end{pmatrix}$, \hspace{1cm} $U = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix}$, \hspace{1cm} $V = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

$^3$ in [26] it is called $JP$-decomposition.
If $A = U_1 - V_1$, where
\[
U_1 = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix},
\]
then there exist the matrices $U_1^{-1}$ and $V_1^{-1}$.
\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]
b) \[
A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},
\]
c) \[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]
If $A = U_1 - V_1$, where
\[
U_1 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 3 & 0 \end{pmatrix},
\]
then there exist the matrices $U_1^{-1}$ and $V_1^{-1}$.

**Remarks 3.3**
1) If for a pair of ordered normed spaces $X = (X, X_+; \|\|)$ and $Y = (Y, Y_+; \|\|)$ with $Y_+ \neq Y$ the wedge $L_+(X, Y)$ is reproducing then each operator in $L(X, Y)$ has a positive decomposition. Moreover, in this case the cone $X_+$ is normal and the cone $Y_+$ is reproducing (see e.g. [3], [5], [25]). As was mentioned in § 2, if $Y$ is a Dedekind complete vector lattice, then the normality of the cones $X_+$ and $Y_+$ and the condition $\text{int}(Y_+) \neq \emptyset$, for example, are sufficient conditions for the wedge $L_+(X, Y)$ to be reproducing.

2) If an operator $A$ has a positive decomposition $A = U - V$ such that $U^{-1}$ exists, then it can be represented as
\[
A = U - V = (I - VU^{-1})U, \quad \text{with} \quad U, V \geq 0. \tag{3.1}
\]

3) The existence of $U^{-1}$ in a positive decomposition of $A$ is also used for the solution of the equation $Ax = y$ by the iterative method $x^{(n+1)} = U^{-1}Vx^{(n)} + U^{-1}y$, which converges for an arbitrary starting vector $x^{(0)}$ provided the spectral radius of $U^{-1}V$ is less than 1.

4) If a decomposition of $A$ is positive regular, then in (3.1) even the operators $U^{-1}$ and $VU^{-1}$ are positive. Therefore, any positive regular decomposition is a $B$-decomposition. The inverse is not true even for matrices (see [21]).

5) The condition $VU^{-1} \geq 0$ is equivalent to the implication $Ux \geq 0 \implies VX \geq 0$. Indeed, assume $VU^{-1} \geq 0$ and let for $x \in X$ be $Ux \geq 0$. Then $(VU^{-1})(Ux) = VX \geq 0$. On the other hand, let $Ux \geq 0$ imply $Vx \geq 0$. If now $x \in X$ then $(UU^{-1})x = x \geq 0$, i.e., with $y = U^{-1}x$ we have $Uy = x \geq 0$ and by assumption $Vy \geq 0$, that is $(VU^{-1})y = Vy \geq 0$.

6) If $A$ decomposes as $A = \lambda I - V$, where $\lambda > 0$ and $V \geq 0$ then the decomposition is positively regular.

7) If $A = \lambda I - V$, $V \geq 0$, $\lambda > 0$ and $A^{-1} \geq 0$ then $\lambda I - V$ is a positive regular decomposition with $r(V) < \lambda$. Indeed, $A^{-1} \geq 0$ implies $(I - \frac{1}{\lambda} V)^{-1} \geq 0$, so by Theorem 2.3 (i) one has $r(I - \frac{1}{\lambda} V) < 1$. 

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8) Let $X = (X, X_+, |||)\] be an ordered Banach space with a normal cone $X_+$. Let $A \in L(X)$ be a positively invertible operator and $\alpha > 0$ a number which satisfies the condition $\alpha \cdot r(A^{-1}) < 1$. If $A \geq -\alpha I$ then $A$ possesses a $B$-decomposition with $r(VU^{-1}) < 1$.

**Proof of statement 8.** Since $(\frac{1}{\alpha} A)^{-1} = \alpha A^{-1}$ and, by assumption $r(\alpha A^{-1}) < 1$, there exists the operator $(I - \alpha A^{-1})^{-1} \geq 0$ and

$$(I - \alpha A^{-1})^{-1} = I + \alpha A^{-1} + \alpha^2 A^{-2} + \alpha^3 A^{-3} + \ldots$$

Put $U = (I - \alpha A^{-1})^{-1} A$ and $V = \alpha A^{-1} U$. Then the assumptions imply that the operator

$$U = A + \alpha A + \alpha^2 A^2 + \alpha^3 A^3 + \ldots$$

is positive and also $V \geq 0$. There exists $U^{-1} = A^{-1}(I - \alpha A^{-1})$ and one has $VU^{-1} = \alpha A^{-1} \geq 0$ and $r(VU^{-1}) = \alpha \cdot r(A^{-1}) < 1$. Moreover,

$$U - V = U - \alpha A^{-1} U = (I - \alpha A^{-1}) U = (I - \alpha A^{-1})(I - \alpha A^{-1})^{-1} A = A.$$

This shows that $U - V$ is a $B$-decomposition of $A$. $\square$

**Theorem 3.4** ([26, Thm. 3]) Let $X = (X, X_+, |||)$ be an ordered Banach space with a normal cone $X_+$ that satisfies the condition $\text{int}(X_+) \neq \emptyset$. Let $A : X \rightarrow X$ be a linear continuous operator. Consider the conditions

(i) $A$ is positively invertible,

(ii) $X_+ \subset A(X_+),$

(iii) there exists some $x_0 \in X_+$ such that $Ax_0 \in \text{int}(X_+),

(iv) $r(VU^{-1}) < 1$ (if $A = U - V$ is a decomposition such that $U^{-1}$ exists).

Then there hold the implications $(i) \Rightarrow (ii) \Rightarrow (iii).$

If $A$ possesses a $B$-decomposition $A = U - V$ then $(iii) \Rightarrow (iv) \Rightarrow (i)$, i.e., in this case all conditions are equivalent.

**Proof.** (i) $\Rightarrow$ (ii). $A^{-1} \geq 0$ means $A^{-1}(X_+) \subset X_+$ and so by applying $A$ to the last inclusion one has $X_+ \subset A(X_+)$.

(ii) $\Rightarrow$ (iii). If now $u \in \text{int}(X_+)$ then $u \in A(X_+)$, i.e., for some $x_0 \in X_+$ one has $Ax_0 = u \in \text{int}(X_+)$.

Let now $A = U - V$ be a $B$-decomposition of $A$. Denote the operator $VU^{-1}$ by $C$. Then the condition b) of a $B$-decomposition implies $C \geq 0$ and according to (3.1) one has the representation $A = (I - C)U$.

(iii) $\Rightarrow$ (iv). From $C \geq 0$ there follows the relation $I - C \leq I$. The positive invertibility of $I - C$ is then guaranteed by Theorem 2.5 if there exists a vector $z_0 \in X_+$ such that $(I - C)z_0 \in \text{int}(X_+)$. By assumption there is a vector $x_0 \in X_+$ with $Ax_0 \in \text{int}(X_+)$. Using (3.1) this yields $(I - C)Ux_0 \in \text{int}(X_+)$. Because $z_0 = Ux_0$ belongs to $X_+$, the cited theorem shows that $I - C$ is positively invertible and so, by Theorem 2.3(i), $r(C) < 1$.

(iv) $\Rightarrow$ (i). The assumptions $r(C) < 1$ and $C \geq 0$ imply that the operator $I - C$ is positively invertible. Then $A$ is invertible as well and from (3.1) there follows that $A^{-1} = U^{-1}(I - C)^{-1}$. We have to establish that $A^{-1} \geq 0$.

This will follow from the implication

$$x \in X, \ Ax \geq 0 \implies x \geq 0. \quad (3.2)$$

Indeed, if $x \geq 0$ then for $z = A^{-1} x$ one has $Az = x \geq 0$ and by the implication (3.2) we get $z \geq 0$ what proves $A^{-1} \geq 0$. For the remaining proof of (3.2) let $x \in X$ and $Ax \geq 0$. Then $(I - C)^{-1}(Ax) = (I - C)^{-1}(I - C)Ux \geq 0$, i.e., $Ux \geq 0$. Property c) of a $B$-decomposition implies now $x \geq 0$.

**Remarks 3.5** 9) If the operator $A$ is invertible then (ii) is a sufficient condition for the positivity of its inverse. Indeed, apply $A^{-1}$ to the inclusion (ii).

10) Only the proof of the implication (iv) $\Rightarrow$ (i) requires the property c) of a $B$-decomposition.
11) Let \( X = (X, X_+, \| \cdot \|) \) be an ordered normed space with a closed normal cone \( X_+ \) and let \( Y = C(Q) \) be the space of all real continuous functions on some compact Hausdorff space \( Q \). If \( A : X \to Y \) is a linear continuous operator then there exist two positive linear continuous operators \( U, V : X \to Y \) such that \( A = U - V \) and
\[
(Ax)_+ \leq Ux \quad \text{and} \quad (Ax)_- \leq Vx \quad \text{for each} \quad x \in X.
\]

This is a part of the proof of [25, Thm. VI.3.2].

Further on we need the fact that the wedge \( L_+(X, Y) \) of all positive linear continuous operators between the ordered normed spaces \( X \) and \( Y \) possesses interior points. Necessary and sufficient conditions were given in [3] (see also [25, Thm. VI.5.1]).

**Theorem 3.6** Let \( (X, X_+, \| \cdot \|) \) and \( (Y, Y_+, \| \cdot \|) \) be ordered normed spaces. For the wedge \( L_+(X, Y) \) to have nonempty interior it is necessary and sufficient that the cone \( X_+ \) allows plastering and the cone \( Y_+ \) has interior points.

If in this case \( f \) is a uniformly positive functional\(^4\) on \( X \) with the positive constant \( \delta \), i.e., \( \delta \|x\| \leq f(x) \) for all \( x \in X_+ \) and \( u \in Y_+ \) with \( B(u, \varepsilon) \subset Y_+ \) for some \( \varepsilon > 0 \), then the ball \( B(f \otimes u, \delta \varepsilon) \) belongs to \( L_+(X, Y) \) and consequently, the rank one operator \( f \otimes u \) is an interior point of \( L_+(X, Y) \). Since any interior point of the cone is an order unit (e.g. [24, Thm. II.1.1]), for any operator \( T \in L(X, Y) \) there is a number \( \lambda > 0 \) such that \( \pm T \leq \lambda(f \otimes u) \).

Now we prove our main result, namely the inverse statement of Theorem 3.4, that the positive invertibility of an operator implies the existence of a \( B \)-decomposition with condition (iv).

**Theorem 3.7** Let \( X = (X, X_+, \| \cdot \|) \) be an ordered Banach space with a (closed) normal cone \( X_+ \) that satisfies the condition \( \text{int}(X_+) \neq \emptyset \) and allows plastering. If an operator \( A \in \mathcal{L}(X) \) is positively invertible then \( A \) possesses a \( B \)-decomposition \( A = U - V \) such that \( r(VU^{-1}) < 1 \).

**Proof.** We suppose that \( A^{-1} \) exists and fulfills \( A^{-1} \geq 0 \). If \( x \in X \) satisfies the condition \( Ax \geq 0 \) then \( A^{-1}(Ax) = x \geq 0 \), i.e., the condition c) of a \( B \)-decomposition in that case automatically holds (even without \( Ux \geq 0 \)).

If \( A \) possesses a \( B \)-decomposition \( A = U - V \) then \( r(VU^{-1}) < 1 \) holds by Theorem 3.4. Therefore it remains to establish the existence of a positive decomposition \( A = U - V \) with the properties a) there exists \( U^{-1} \) and b) \( VU^{-1} \geq 0 \).

This can be done in the following way: Let \( T \geq 0 \) be an arbitrary non-zero (positive) operator such that \( r(T) < 1 \). In this case the operator \( (I - T)^{-1} \) exists and, according to Theorem 2.3(i), satisfies the condition \( (I - T)^{-1} \geq 0 \). Put
\[
U = (I - T)^{-1}A \quad \text{and} \quad V = TU.
\]

Then there exists also the operator \( U^{-1} \), where \( U^{-1} = A^{-1}(I - T) \). The operator \( VU^{-1} \) is positive, because of \( VU^{-1} = TUU^{-1} = T \geq 0 \). Moreover,
\[
U - V = (I - T)^{-1}A - TU = (I - T)^{-1}A - T(I - T)^{-1}A = (I - T)(I - T)^{-1}A = A.
\]

If now \( U \geq 0 \) then also \( V \geq 0 \) holds, and \( U - V \) is the required \( B \)-decomposition of \( A \). Consequently, the only fact to be proved is that \( U \geq 0 \). We can do this if the operator \( T \) is constructed appropriately. Let \( u \) be a fixed interior point of the cone \( X_+ \) and \( f \) a uniformly positive functional on \( X \). It is clear that the operator \( f \otimes u \) is positive and \( f(A^{-1}u) > 0 \). It was already mentioned that the operator \( f \otimes u \) is an interior point of the wedge \( L_+(X) \). According to the remark after Theorem 3.6 there is some \( \lambda > 0 \) such that \( -A \leq \lambda(f \otimes u) \). Define for each \( \alpha > 0 \) the operator
\[
T_\alpha = \frac{1}{f(A^{-1}u) + \alpha} (f \otimes u) A^{-1}.
\] \( \tag{3.3} \)

Then \( T_\alpha \geq 0 \) and a simple calculation shows that \( T_\alpha^{n+1} = q^n T_\alpha \), where \( q = \frac{f(A^{-1}u)}{f(A^{-1}u) + \alpha} \) and
\[
r(T_\alpha) = \lim_{n \to \infty} \sqrt[n]{q^n \|T_\alpha\|} = q \lim_{n \to \infty} \sqrt[n]{\|T_\alpha\|} = q < 1 \quad \text{for all} \quad \alpha > 0.
\]

\(^4\) Such a functional exists due to the cone allows plastering, see [25].
So, due to \( r(T_\alpha) < 1 \) for all \( \alpha > 0 \) the operator \((I - T_\alpha)^{-1}\) exists and is positive. Therefore

\[
(I - T_\alpha)^{-1} = I + \sum_{n=0}^{\infty} q^n T_\alpha = I + \frac{f(A^{-1}u) + \alpha}{\alpha} T_\alpha.
\]

The operator \((I - T_\alpha)^{-1}A\) has now the representation

\[
(I - T_\alpha)^{-1}A = A + \frac{f(A^{-1}u) + \alpha}{\alpha} T_\alpha A = A + \frac{1}{\alpha} (f \otimes u).
\]

If \( \alpha \) satisfies the inequality \( \lambda \alpha \leq 1 \) then \((I - T_\alpha)^{-1}A \geq A + \lambda (f \otimes u) \geq 0\) and the operator \( U = (I - T_\alpha)^{-1}A \) is positive. This completes the proof according to the argument at the beginning of the proof. \( \square \)

**Examples 3.8**

1. For a square matrix \( A \) one obtains now the following result of J. E. Peris ([21, Theorem 5]), where the underlying space is \((\mathbb{R}^n, \mathbb{R}_+^n, \|\cdot\|)\) considered with an arbitrary norm.

   A square matrix \( A \) is positively invertible if and only if \( A \) allows a B-decomposition \( A = U - V \) with \( r(VU^{-1}) < 1 \).

   **Proof.** Due to Theorem 3.4 only the necessity has to be proved. If \( A = (a_{ij}) \) is an \( n \times n \)-matrix and \( u = f = (1, \ldots, 1) \) then \( u \) is an interior point of the cone \( \mathbb{R}_+^n \) and \( f \) is a uniformly positive functional\(^5\) on \( \mathbb{R}^n \) such that \( f(A^{-1}u) > 0 \). Then the operator \( T_\alpha \), defined in the proof of Theorem 3.7, is positive and satisfies \( r(T_\alpha) < 1 \). Some direct calculation yields

\[
(I - T_\alpha)^{-1}A = A + \frac{1}{\alpha} (f \otimes u),
\]

where \( f \otimes u \) is the matrix \( E = (e_{ij}) \) with \( e_{ij} = 1 \) for all \( i, j = 1, \ldots, n \). Again for sufficiently small \( \alpha \) the matrix \((I - T_\alpha)^{-1}A\) is positive and so the required decomposition can be constructed as the proof of the preceding theorem indicates. \( \square \)

2. For the real Banach space \( c_0 \) of all null sequences let

\[
E = \mathbb{R} \oplus c_0 = \left\{ \left( \xi \atop x \right) : \xi \in \mathbb{R}, x = (x_k)_{k \in \mathbb{N}} \in c_0 \right\}
\]

be its one-dimensional extension with the norm \( \| \left( \xi \atop x \right) \| = \sqrt{\xi^2 + \|x\|^2} \). A linear continuous operator \( B \) in \( E \) can be described as a matrix

\[
B = \begin{pmatrix} \eta & \phi \\ z & C \end{pmatrix},
\]

where \( \eta \in \mathbb{R}, z \in c_0, \phi \) is linear continuous functional on \( c_0 \) and \( C \) is a linear continuous operator in \( c_0 \) (see \([20]^6\)). The operator \( B \) acts on an element \( \left( \xi \atop x \right) \in E \) as

\[
B \left( \xi \atop x \right) = \left( \xi \eta + \phi(x) \atop \xi z + Cx \right).
\]

The subset in \( E \)

\[
E_+ = \left\{ \left( \xi \atop x \right) : \xi \geq \|x\| = \sup_{k \in \mathbb{N}} |x_k| \right\}
\]

\(^5\) due to \( \|x\| = (x_1^2 + \cdots + x_n^2)^{1/2} \leq x_1 + \cdots + x_n = f(x) \) for any \( x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \), if \( \|\cdot\| \) is the Euclidean norm.

\(^6\) The author thanks Prof. K. C. Sivakumar, IIT Madras (Chennai), for bringing this paper to his attention.
is known as the ice cream cone\textsuperscript{7} in $E$. $E_+$ is a closed normal cone with nonempty interior and a norm bounded base. The latter can be seen since $\xi \geq ||x||$ is equivalent to $\xi \geq \frac{1}{\gamma_2}\|\xi\|$. The linear continuous functional $f : E \to \mathbb{R}$ defined by $f(\xi) = \xi$ is uniformly positive and therefore $E_+$ allows plastering. Moreover, $||f|| = 1$.

For a slightly more general construction and for properties of ice cream cones we refer also to [2]. A linear operator $B$ in $E$ is positive with respect to the cone $E_+$, i.e., $B(E_+) \subset E_+$, if and only if

$$ \sup_{||x|| = 1} (||Cx + z|| - \phi(x)) \leq \eta $$

holds (see [20, Lemma 2]). There, in particular, it is shown that an operator $B$ with

$$ z = (z_k)_{k \in \mathbb{N}} , \quad \phi = (-\phi_k)_{k \in \mathbb{N}} \in \ell_1 , \quad \eta = 1 + \sum_{k \in \mathbb{N}} \phi_k \quad \text{and} \quad C = \text{diag}(1 - z_k)_{k \in \mathbb{N}} $$

(3.4)

is positive, where $0 < z_k < 1$, $z_k \neq z_j$ for $k \neq j$, $\phi_k > 0$ and $\sum_{k \in \mathbb{N}} \frac{\phi_k}{z_k} = 1$ are assumed. We select $z = (z_k)_{k \in \mathbb{N}} \in c_0$ such that $\sum_{k \in \mathbb{N}} \frac{z_k}{1 - z_k} < +\infty$. Observe that the operator $C$ is invertible on $c_0$.

If now

$$ B \left( \begin{array}{c} \xi \\ x \end{array} \right) = 0 \quad \text{for some} \quad \left( \begin{array}{c} \xi \\ x \end{array} \right) $$

then $\xi z + Cx = 0$ and $\xi \eta + \phi(x) = 0$. The first equality is equivalent to $\xi z_k + x_k - z_k x_k = 0$ for all $k \in \mathbb{N}$, what yields

$$ x_k = -\frac{z_k - \xi}{1 - z_k} . $$

(3.5)

Substitute $x_k$ and $\eta$ into the equality $\xi \eta + \phi(x) = 0$ one obtains

$$ 0 = \xi \eta - \sum_{k \in \mathbb{N}} \phi_k x_k = \left( 1 + \sum_{k \in \mathbb{N}} \frac{\phi_k z_k}{1 - z_k} \right) \xi , $$

The convergence of the last series implies $\xi = 0$. Then due to (3.5) also $x_k = 0$ for all $k \in \mathbb{N}$. Consequently, $(\xi) = 0$ and so, $B$ is invertible.

The inverse operator of $B$ can be written as well as a matrix

$$ B^{-1} = \begin{pmatrix} \omega & \psi \\ u & S \end{pmatrix} $$

with $\omega \in \mathbb{R}$, $u \in c_0$, $\psi \in \ell_1$ and $S$ a linear continuous operator on $c_0$. Then $BB^{-1} = B^{-1}B = I$ yield the following relations

\begin{align*}
(a) \quad \eta \omega + \phi(u) &= 1, & (a') \quad \eta \omega + \psi(z) &= 1, \\
(b) \quad \omega z + Cu &= 0, & (b') \quad \eta u + Sz &= 0, \\
(c) \quad \eta \psi + \phi \circ S &= 0, & (c') \quad \omega \phi + \psi \circ C &= 0, \\
(d) \quad \psi \otimes z + CS &= I_{c_0}, & (d') \quad \phi \otimes u + SC &= I_{c_0},
\end{align*}

where $I_{c_0}$ is the identity operator in $c_0$. It is a routine to calculate now the elements in the representation of the operator $B^{-1}$ when $B$ is the operator with the data given in (3.3). Namely, we obtain

$$ \omega = \left( 1 + ||\phi|| + \sum_{k = 1}^{\infty} \frac{\phi_k z_k}{1 - z_k} \right)^{-1} , \quad \psi = -\omega C^{-1} \phi , \quad u = -\omega C^{-1} z , \quad S = C^{-1} - \psi \otimes C^{-1} z. $$

\textsuperscript{7} in [20] this cone is called hyperbolic.
Consider now the operator \( A = B^{-1} \). Then \( A \) is a positively invertible operator in the ordered Banach space \( E \), where the cone \( E_+ \) satisfies all conditions of Theorem 3.7. Consequently, \( A \) has a \( B \)-decomposition \( A = U - V \) with \( r(VU^{-1}) < 1 \). The rank one operator \( f \otimes \left( \begin{array}{cc} 1 \\ 0 \end{array} \right) \) is an order unit in \( L_+(E) \) since \( \left( \begin{array}{cc} 1 \\ 0 \end{array} \right) \) is an inner point of the cone \( E_+ \) and \( f \) a uniformly positive functional on \( E \). There exists a number \( \lambda > 0 \) such that \( -\lambda \leq \lambda \left( f \otimes \left( \begin{array}{cc} 1 \\ 0 \end{array} \right) \right) \). Let be

\[
T = \frac{1}{\eta + \lambda} \left( f \otimes \left( \begin{array}{cc} 1 \\ 0 \end{array} \right) \right) A^{-1} = \frac{\lambda}{1 + \lambda \eta} \left( \begin{array}{cc} \eta & \phi \\ 0 & 0 \end{array} \right).
\]

Then \( T \geq 0 \) and, according to Theorem 3.7, the following decomposition of the operator \( A \) is established:

\[
A = U - V, \quad \text{where} \quad U = (I - T)^{-1}A = A + \lambda \left( f \otimes \left( \begin{array}{cc} 1 \\ 0 \end{array} \right) \right) \quad \text{and} \quad V = T(I - T)^{-1}A.
\]

More detailed, one calculates the very simple operators

\[
U = \left( \begin{array}{cc} \omega + \lambda & \psi \\ u & S \end{array} \right) \quad \text{and} \quad V = \left( \begin{array}{cc} \lambda & 0 \\ 0 & 0 \end{array} \right),
\]

showing also the apparent meaning of \( \lambda \) in the operator \( V \). This, of course, is caused by the simple structure of the interior point \( f \otimes \left( \begin{array}{cc} 1 \\ 0 \end{array} \right) \) in \( L_+(E) \).

Remarks 3.9

11) The decomposition of an operator can be reformulated in terms of its majorization (see [19]). Then the existence of a \( B \)-decomposition for an operator \( A \) in an ordered normed space is equivalent to the existence of an invertible majorant \( U \) such that \( I - AU^{-1} \) is a positive operator with the spectral radius less than 1.

12) The question formulated in [27] is answered confirmatively, i.e., under the conditions of Theorem 3.7 all statements of Theorem 3.4 hold, in particular, each invertible operator \( A \) possesses a \( B \)-decomposition \( A = U - V \) with \( r(VU^{-1}) < 1 \).

13) The proof of Theorem 3.7 shows that namely the operators \( U = A + \frac{1}{\alpha} (f \otimes u) \) and \( V = \frac{1}{\alpha} (f \otimes u) \) form a \( B \)-decomposition of \( A \). Their construction essentially uses the order unit \( f \otimes u \) in \( L_+(E) \). In Example 2. for \( \frac{1}{\alpha} \) was taken the number \( \lambda \).

14) An open question is, whether a positively invertible operator in an ordered Banach space \( E \) can have a \( B \)-decomposition, when \( L_+(E) \) does not possess any order unit, in particular, if the condition \( \text{int}(X) \neq \emptyset \) fails in Theorem 3.7 or is replaced by another one.

References