GLOBAL INTEGRABILITY OF SUPERTEMPERATURES ON A JOHN CYLINDER

HIROAKI AIKAWA

ABSTRACT

Ever since Armitage showed that every nonnegative superharmonic function on a bounded domain of bounded curvature (= $C^{1,1}$ domain) in $\mathbb{R}^n$ is $L^p$-integrable up to the boundary for $0 < p < n/(n-1)$, the global integrability of nonnegative supersolutions has attracted many mathematicians.

In this talk we consider a parabolic counterpart. We study the global integrability of nonnegative supertemperatures on the cylinder $D \times (0, T)$, where $D$ is a Lipschitz domain or a John domain. We show that the integrability depends on the lower estimate of the Green function for the Dirichlet Laplacian on $D$.

In particular, if $D$ is a bounded $C^1$-domain, then every nonnegative supertemperature on $D \times (0, T)$ is $L^p$-integrable over $D \times (0, T')$ for any $0 < T' < T$, provided $0 < p < (n+2)/(n+1)$. The bound $(n+2)/(n+1)$ is sharp.

We employ different arguments for a Lipschitz cylinder and for a John cylinder. While heat kernel estimates are crucial for a Lipschitz cylinder, they are not available for a John cylinder. A parabolic box argument related to intrinsic ultracontractivity plays a crucial role instead.

Joint work with Hara and Hirata.

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Progressive intrinsic ultracontractivity for nonlocal Schrödinger operators

We give sharp two-sided large time estimates of the heat kernel for a large class of non-local Schrödinger operators with confining potentials, which are based on generators of Lévy processes. We identify a new useful regularity property of compact semigroups, which is weaker than asymptotic intrinsic ultracontractivity. It means that the space-time regularity of the semigroup essentially improves as the time parameter diverges to infinity. This is a joint work with René Schilling.

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Hilbert spaces and reproducing kernels for parabolic operators of fractional order*†‡

Masaharu Nishio§

Hardy space and Bergman space on the unit disc in the complex plane are well-know Hilbert spaces, which have the reproducing kernels. They are function spaces which consist of holomorphic functions.

In this talk, I consider a parabolic operator $L^{(a)} := \partial_t + (-\Delta_x)^a$ and its iterates on the upper half space $H := \{(x,t)| x \in \mathbb{R}^n, t > 0\}$, and discuss the analogue of the above Bergman space. I also mention the relation of harmonic and polyharmonic functions with the Poisson operator $L^{(1/2)}$.

Fundamental properties of the harmonic Bergman space on the upper half space are discussed in [5], and my talk is based on results in [1, 2, 3, 4].

References


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Compactness of Markov and Schrödinger semi-groups

Yoshihiro Tawara (jointwork with M. Takeda and K. Tsuchida)

Let $E$ be a locally compact separable metric space and $m$ a positive Radon measure on $E$ with full support. Let $X$ be an $m$-symmetric Markov process on $E$. We assume that $X$ is irreducible and has strong (resolvent) Feller property. Moreover, we assume that $X$ possesses the tightness property, i.e., for any $\epsilon > 0$ there exists a compact set $K$ such that $\sup_{x \in E} R_1 1_{K^c}(x) \leq \epsilon$. Here $R_1$ is the 1-resolvent of $X$ and $1_{K^c}$ is the indicator function of the complement of $K$. When $X$ has these properties, we say in this talk that $X$ belongs to Class (T). Takeda proved that if $X$ belongs to Class (T), its semi-group turns out to be a compact operator on $L^2(E;m)$. In this talk, we apply this criterion to Dirichlet Laplacians $\Delta_D$ and Schrödinger operators $\Delta - V$ with positive potential and show probabilistically the compactness of these operators.
Periodic observations of spectrally one-sided Lévy processes and applications to inventory control

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Abstract

We consider a version of the stochastic inventory control problem for a spectrally positive Lévy demand process, in which the inventory can only be replenished at independent exponential times. We compute, via the scale function, the fluctuation identities for the controlled process under a periodic barrier policy, which replenishes any shortage below a certain barrier at each replenishment opportunity. These identities can be used to show the optimality of such policy in a straightforward manner. Numerical results are also provided. This is based on a joint work with A. Bensoussan and J.L. Pérez.

Keywords: inventory models; spectrally one-sided Lévy processes; scale functions; periodic observation; resolvents

References


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On the construction of a general stable-like Markov process

Victoria Knopova (TU Dresden)

We consider an integro-differential operator

\[ Lf(x) = b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d \setminus \{0\}} \left( f(x + u) - f(x) - \nabla f(x) \cdot u 1_{|u| \leq 1} \right) N(x, du), \]

defined on the space \( C^2_{\infty}(\mathbb{R}^d) \) of twice continuously differentiable functions with vanishing at infinity derivatives. The drift \( b \in \mathbb{R}^d \) is assumed to be bounded and Hölder continuous, and the Lévy-type kernel \( N(x, du) \) is a sum of an \( \alpha \)-stable like part and a lower order perturbation.

We show that under certain regularity assumptions the extension of \( (L, C^2_{\infty}(\mathbb{R}^d)) \) is the generator of a Feller semigroup \( (P_t)_{t \geq 0} \).

The talk is based on the on-going work with A. Kulik and R. Schilling.
Generalized arcsine laws
for null recurrent diffusions
and for infinite ergodic transformations

Kouji YANO (Kyoto University)

For the simple symmetric random walk \( \{X_n\} \) on \( \mathbb{Z} \), Lévy’s arcsine law \([3]\) asserts that

\[
\frac{1}{n} \sum_{k=0}^{n-1} 1_{\{X_k > 0\}} \xrightarrow[n \to \infty]{d} \frac{d}{\pi \sqrt{x(1-x)}} \text{ on } (0,1),
\]

whose limit is called the arcsine distribution. There have been lots of studies generalizing this result from various aspects.

In this talk we focus on generalizations of Lamperti type. We recall several results of generalizations for null recurrent diffusions. We also recall similar results for infinite ergodic transformations and then we shall mention our recent work \([4]\), jointly with Toru Sera, about generalization of previous results by utilizing the methods which have been employed for diffusions.

Lamperti \([2]\) studied a Markov chain \( \{X_n\} \) on \( \mathbb{Z} \) which cannot skip 0 when it changes signs and proved that the convergence

\[
\frac{1}{n} \sum_{k=0}^{n-1} 1_{\{X_k > 0\}} \xrightarrow[n \to \infty]{d} Z
\]

holds for some non-trivial random variable \( Z \) only if \( Z \) follows the Lamperti distribution with parameters \( \alpha, p \in (0,1) \):

\[
\frac{\sin \alpha \pi}{\pi} \cdot \frac{p(1-p)x^{\alpha-1}(1-x)^{\alpha-1}dx}{p^2(1-x)^{2\alpha} + (1-p)^2x^{2\alpha} + 2p(1-p)x^\alpha(1-x)^\alpha \cos \alpha \pi} \text{ on } (0,1). \tag{3}
\]

Barlow–Pitman–Yor \([1]\) studied the class of skew Bessel diffusions on multiray, i.e., a finite number of half lines called rays radiating from the origin, and obtained multiray generalization of the Lamperti distributions. These results have been generalized to null recurrent diffusions by Truman–Williams \([7]\), Watanabe \([8]\), Watanabe–K. Yano–Y. Yano \([9]\) and Y. Yano \([10]\).

Thaler \([5]\) studied a class of infinite ergodic transformations \( T \) where the state space is divided into a disjoint union \( A_1 + Y + A_2 \) and the orbit \( \{T^na\} \) of an initial state \( a \) cannot skip the junction \( Y \) when it changes rays \( A_1 \) and \( A_2 \). A typical example is the Boole transform defined by

\[
Ta = a - \frac{1}{a} \quad (a \in \mathbb{R} \setminus \{0\}), \quad T0 = 0, \tag{4}
\]

which is an ergodic transformation preserving the Lebesgue measure \( \lambda \). Note that the orbit of any initial state cannot skip the junction \( Y = [-1, 1] \) when it changes rays \( (-\infty, -1] \).
and $[1, \infty)$. He proved that, for any probability measure $\nu$ absolutely continuous w.r.t. $\lambda$, the invariant measure, the following convergence holds:

$$
\frac{1}{n} \sum_{k=0}^{n-1} 1_{\{T^k a > 0\}} \text{(under $\nu(da)$)} \xrightarrow{d} \frac{dx}{\pi \sqrt{x(1-x)}} \text{ on } (0, 1),
$$

where we note that

$$
\frac{1}{n} \sum_{k=0}^{n-1} 1_{\{T^k a \in Y\}} \text{(under $\nu(da)$)} \xrightarrow{a.s.} 0.
$$

The Boole transform can be deformed into an interval map having two indifferent fixed points. He obtained a limit theorem of Lamperti type for infinite ergodic interval maps $T$ assuming certain regular variation conditions on $T$ at the two indifferent fixed points of $T$. His theory was developed by Thaler–Zweimüller [6] and Zweimüller [11].

Our recent results of Sera–K. Yano [4] is a multiray generalization of the previous results [5], [6] and [11]. Their method was convergence of moments, which seems quite difficult to handle the joint distribution of occupation ratios. Our method is based on the double Laplace transforms, which have played a crucial role in the above-mentioned papers about null recurrent diffusions.

References


A mechanical model of Brownian motion

Song Liang (University of Tsukuba)

We provide a connection between Brownian motion and a classical mechanical system. Precisely, we consider a system of one massive particle interacting with an idea gas, evolved according to non-random Newton mechanical principle, via a repulsive interaction potential function, and prove that under certain condition, the (position, velocity)-process of the massive particle converges to a diffusion under a certain scaling limit, such that the mass of the light particles converges to 0, while the density and the velocities of them go to infinity.
Coagulation-fragmentation equations and underlying stochastic dynamics  
Kenji Handa (Saga U)

We consider stochastic dynamics of interval partitions evolving according to certain split-merge transformations. An asymptotic result for properly rescaled processes is shown to obtain a solution to a nonlinear equation called the coagulation-fragmentation equation.
Let \((X, d)\) be a proper ultrametric space. Given a measure \(m\) on \(X\) and a function \(B \mapsto C(B)\) defined on the set of all non-singleton balls \(B \subset X\) we consider the hierarchical Laplacian
\[
L_C f(x) := \sum_{B \in B : x \in B} C(B) \left( f(x) - \frac{1}{m(B)} \int_B f \, dm \right).
\]
Choosing a sequence \(\{\varepsilon(B)\}\) of i.i.d. random variables we define the perturbed function \(C(B, \omega)\) and the perturbed hierarchical Laplacian \(L^\omega = L_{C(\omega)}\). In the talk we discuss convergence of the sequence of arithmetic means of the \(L^\omega\)-eigenvalues to a normal distribution. We also study related point processes built of the eigenvalues and their convergence to a Poisson point process.

Based on a joint project with A. Bendikov (Univ. Wrocław).
Another View of the Riccatti equations arising in Affine Class in Finance

JIRO AKAHORI (RITSUMEIKAN U)

Fourier transform of the transition probability of (some) processes in “affine class” is given by solving Riccatti type equation. Recently, a “fractional” extension of this well known fact in quantitative finance are found in the context of ”rough volatility”. Inspired by the result, I studied the problem in an infinite-dimensional way to see the equation in a different way.
A Discrete Scheme of Static Hedging of Barrier Options

Yuri Imamura *

Abstract

We consider a discrete scheme for static hedging of barrier options, by establishing a discrete version of the transform which Peter Carr and Sergey Nadtochiy (2013) introduced, for a general one dimensional diffusion case. The transform describes the (put type) pay-off which balances at the barrier with a given (call-type) pay-off and hence the plain option with the former pay-off statically hedges a barrier option with the latter pay-off. In this talk I will construct the map for a class of Markov chains, which includes a discretization of Carr-Nadtochiy’s correspondence, and also its multi-dimensional version. The latter gives a new insight to the literature.

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A fractional calculus approach to rough path integration

Yu Ito (Kyoto Sangyo U)

This study is an alternative approach to the fundamental theory of rough path analysis on the basis of fractional calculus. In this talk, using fractional calculus, we will provide alternative expressions of the rough path integrals by T. J. Lyons (1998) and by M. Gubinelli (2004). The expressions are given explicitly by the Lebesgue integrals for fractional derivatives. Our results can be regarded as a generalization of those of Y. Hu and D. Nualart (2009), and one of the key ingredients for our results is a method by M. Zähle (1998).
Convergence of diffusion processes in a tube and Dirichlet forms of limit processes

Tomoko TAKEMURA

July 6, 2018

1 Preliminary

Let $s$ be a continuous increasing function on an open interval $(r_1, r_2)$, where $-\infty \leq r_1 < r_2 \leq \infty$, $m$ be a right continuous increasing function on $(r_1, r_2)$. If $r_i$ is $(s, m)$-regular in the sense of Feller, then absorbing or reflecting boundary condition is posed at $r_i$. Let $\mathcal{G}_{s,m}$ be a one dimensional generalized diffusion operator on $(r_1, r_2)$ with a scale function $s$ and a speed measure $m$. We denote by $X_{s,m} = [X(t), P^X_t]$ on $(r_1, r_2)$ the one dimensional diffusion process on $(r_1, r_2)$ whose generator is given by $\mathcal{G}_{s,m}$. Here we consider $X^{(i)} = X_{s^{(i)}, m^{(i)}}$ on $(r_1, r_2)$, $R_n = X_{s^n, m^n}$ on $I_n = (0, l_n)$, and $R = X_{s,m}$ on $I = (0, l)$, where $-\infty \leq r_1 < r_2 \leq \infty$ and $0 < l_n$, $l \leq \infty$ for $n \in \mathbb{N}$. We assume $|s^{(i)}(r_i)| = \infty$, $i = 1, 2$, the left end point $0$ is $(s_n, m_n)$-entrance and $(s, m)$-entrance, and the absorbing or reflecting boundary condition is posed at $l_n$\text{[resp. $I$]} whenever it is $(s_n, m_n)$-regular\text{[resp. $(s, m)$-regular]}. Let $\Theta = [\Theta(t), P^\Theta_t]$ be a spherical Brownian motion on $S^{d-1}$.

Let $\nu_n$ be a Radon measure on $I_n$ and assume that the support of $\nu_n$ coincides with $I_n$, $|\int_{(0,c)} s_n(x) \, dv_n(r)| = \infty$, $\forall c \in I_n$, $\forall n \in \mathbb{N}$, and $|\int_{(c,l_n)} dv_n(r)| < \infty$ if $l_n$ is $(s_n, m_n)$-regular and reflecting. We set $\nu$ and make the assumptions for $\nu$ as well as $\nu_n$. We set

$$f_n(t) = \int_{I_n} l^R_n(t,r) \, dv_n(r), \quad f(t) = \int_I l^R(t,r) \, dv(r), \quad t \geq 0,$$

where $l^R_n(t,r)$\text{[resp. $l^R(t,r)$]} is the local time of $R_n$\text{[resp. $R$]}. Let

$$Y_n = \left[ Y_n(t) = \left( R_n(t), \Theta(f_n(t)), X^{(1)}(t), P^Y_n(r,\theta,x) \right), Y(t) = \left( R(t), \Theta(f(t)), X^{(1)}(t), P^Y(r,\theta,x) \right) \right],$$

where $P^Y_n(r,\theta,x) = P^R_n \otimes P^\Theta \otimes P^X_n(r,\theta,x)$, $(r, \theta, x) \in I_n \times S^{d-1} \times (r_1, r_2)$ and $P^Y(r,\theta,x) = P^R \otimes P^\Theta \otimes P^X(r,\theta,x)$, $(r, \theta, x) \in I \times S^{d-1} \times (r_1, r_2)$. We note that $(R(t), \Theta(f(t)))$ is the skew product $\Xi$ of the one dimensional additive diffusion process $R$ and the spherical Brownian motion $\Theta$ with respect to the positive continuous functional $f(t)$. Next we consider the time changed processes. We set

$$\Psi_n(t) = \int_0^t \mu_{1,n}(R_n(s)) \mu_{2,n}(\Theta(f(s))) \, ds, \quad \Psi(t) = \int_I l^R(t,r) \, d\mu(r), \quad t > 0,$$

where $\mu_{1,n}(r)\mu_{2,n}(\theta)$ is a bounded measurable function on $I_n \times S^{d-1}$ which is bounded below by a positive constant and $d\mu(r) = 1_\Lambda(r) \, dm(r)$ for $\Lambda \subset I$. We set $\Gamma = \Lambda \times S^{d-1} \times (r_1, r_2)$. We denote by $\Phi_n(t)$\text{[resp. $\Phi(t)$]} the right continuous inverse of $\Psi_n(t)$\text{[resp. $\Psi(t)$]}. We consider the time changed processes

$$\Xi_n = \left[ \Xi_n(t) = Y_n(\Phi_n(t)), P^\Xi_n \right], \quad \Xi = \left[ \Xi(t) = Y(\Phi(t)), P^\Xi \right].$$
2 Main theorem

Theorem 1 Under some assumptions, the time changed process $\mathbb{X}_n$ converge to the time changed process $\mathbb{X}$ in the following sense.

$$\lim_{n \to \infty} p_{X_n}^X(r, \theta, x) = p_{X}^X(r, \theta, x)$$

for $t > 0$, $(r, \theta, x) \in \Gamma$, and $f \in C_b((0, \infty) \times S^{d-1} \times (r_1, r_2))$, where the semi group $p_{X}^X(r, \theta, x) = E_{P_{X}^X(r, \theta, x)}[f(X_t)]$, and $\tilde{\mathbb{X}}_n[resp. \tilde{\mathbb{X}}]$ associate with $\tilde{s}_n$, $\tilde{m}_n$, $\tilde{v}_n$, and $\tilde{\mu}_n, \tilde{\mu}$.

Let $(\mathcal{E}^\theta, \mathcal{F}^\theta)$ be a Dirichlet form on $L^2(S^{d-1}, \mu^\theta)$ corresponding to $\Theta$ on $S^{d-1}$, and $(\mathcal{E}^D, \mathcal{F}^D)$ be a Dirichlet form on $L^2((r_1, r_2), m^D)$ corresponding to $X^{(1)}$. We denote by $S \times D = S^{d-1} \times S^{d-1} \times (r_1, r_2) \times (r_1, r_2)$ and $\mathcal{M}$ the product measure $m^\Theta \otimes m^\Theta \otimes m^D \otimes m^D$. We note that $I \setminus \Lambda = \bigcup_{k \in K} I_k$, a finite or a countable disjoint union of open intervals $I_k = (a_k, b_k)$ with the end points belonging to $\Lambda \cup \{0, 1\}$.

Theorem 2 Assume $\Lambda \neq I$, then the Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$ of $\mathbb{X}$ is regular on $L^2(\Gamma, \mu \otimes m^\Theta \otimes m^D)$ and has $C^2[\Gamma]$ as a core. For $f \in C^2[\Gamma]$, the Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$ is given by the following.

$$\mathcal{E}^X(f, f) = \int_{\Gamma} \partial^2_{\theta^2} f(r, \theta, x)^2 \, dm^R(r) \, dm^\Theta(\theta) \, dm^D(x) + \int_{\Lambda \times (r_1, r_2)} \mathcal{E}^\Theta(f(r, \cdot, x), f(r, \cdot, x)) \, d\nu(r) \, dm^D(x)$$

$$\quad + \int_{\Lambda \times S^{d-1}} \mathcal{E}^D(f(r, \theta, \cdot), f(r, \theta, \cdot)) \, dm^R(r) \, dm^\Theta(\theta)$$

$$\quad + \frac{1}{2} \sum_{*1} \int_{S \times D} \{f(a_k, \zeta) - f(a_k, \eta)\}^2 J_1^{R_k}(\zeta, \eta : a_k, a_k) \, d\mathcal{M}(\zeta, \eta)$$

$$\quad + \frac{1}{2} \sum_{*2} \int_{S \times D} \{f(b_k, \zeta) - f(b_k, \eta)\}^2 J_2^{R_k}(\zeta, \eta : b_k, b_k) \, d\mathcal{M}(\zeta, \eta)$$

$$\quad + \frac{1}{2} \sum_{*3} \int_{S \times D} \{f(a_k, \zeta) - f(b_k, \eta)\}^2 J_3^{R_k}(\zeta, \eta : a_k, b_k) \, d\mathcal{M}(\zeta, \eta)$$

$$\quad + \frac{1}{2} \sum_{*3} \int_{S \times D} \{f(b_k, \zeta) - f(a_k, \eta)\}^2 J_4^{R_k}(\zeta, \eta : b_k, a_k) \, d\mathcal{M}(\zeta, \eta) + I(f), \quad (1)$$

where $*1 = k \in K$, $0 \leq a_k < b_k < l$, $*2 = k \in K$, $0 < a_k < b_k \leq l$, and $*3 = k \in K$, $0 < a_k < b_k < l$.

Here the last term $I(f)$ should be read as

$$\frac{1}{s^R(l) - s^R(a_k)} \int_{S^{d-1} \times (r_1, r_2)} f(a_k, \theta, x) \, dm^\Theta(\theta) \, dm^D(x)$$

if $0 < a_k < b_k = l$, $l$ is $(s^R, m^R)$-regular with absorbing, -exit, or -natural and $s^R(l) < \infty$. $I(f) = 0$ if $0 < a_k < b_k = l$ and $s^R(l) = \infty$. Furthermore the first term of the right hand side of $(1)$ vanishes in case that $\int_{\Lambda} ds^R(r) = 0$.

In this talk we show details of assumptions and proofs and give the jumping measures $J_i^{R_k}(\zeta, \eta : a_k, b_k)$, $i = 1, 2, 3, 4$ and the killing part $I(f)$. 
Distance Multivariance - measuring and detecting multivariate dependence

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We introduce distance multivariance $M$ and related quantities and highlight some properties. The talk is based on the recent preprints [1,2,3,4].

Let $X_1, \ldots, X_n$ be random vectors with possibly distinct dimensions, i.e., with values in $\mathbb{R}^{d_i}$ for $i = 1, \ldots, n$. If these vectors are $(n-1)$-independent then $M(X_1, \ldots, X_n) = 0$ characterizes the independence of $X_1, \ldots, X_n$. This is the basis for the construction of explicit measures of (in)dependence, e.g. total distance multivariance $\overline{M}$ and $m$-distance multivariance $M_m$ such that

$$\overline{M}(X_1, \ldots, X_n) = 0 \text{ if and only if } X_1, \ldots, X_n \text{ are independent},$$

$$M_2(X_1, \ldots, X_n) = 0 \text{ if and only if } X_1, \ldots, X_n \text{ are pairwise independent}.$$

In addition to the theoretical characterization there exist corresponding sample versions. These are computational efficient and their distributional properties are known. Thus empirical measures and empirical tests of (in)dependence can be constructed and implemented. All functions for the application of distance multivariance are published in the R package multivariance [5].

Roughly speaking, distance multivariance is constructed as the distance of characteristic functions in an $L^2$-space with respect to symmetric Lévy measures, and it has an equivalent representation in terms of expectations of continuous negative definite functions applied to differences of random variables. This yields that the sample version can be calculated using (doubly centered) distance matrices of the samples.

References

Density for the solution to stochastic functional differential equations

Atsushi TAKEUCHI
(Osaka City University, JAPAN)

Let $T$ and $r$ be constant fixed throughout the talk, and $A : [0, T] \times C([-r, 0]; \mathbb{R}^d) \to \mathbb{R}^d$ and $B : [0, T] \times C([-r, 0]; \mathbb{R}^m \otimes \mathbb{R}^d)$, with certain nice conditions on the regularity and the boundedness. For a deterministic path $\eta \in C([-r, 0]; \mathbb{R}^d)$, we shall consider stochastic functional differential equations of the form:

$$X(t) = \begin{cases} 
\eta(t) & (-r \leq t \leq 0), \\
\eta(0) + \int_0^t A(s, X_s) \, ds + \int_0^t B(s, X_s) \, dW(s) & (0 < t \leq T),
\end{cases}$$

where $W = \{W(t) : 0 \leq t \leq T\}$ is the $m$-dimensional Brownian motion starting at the origin, and $X_t = \{X(t + u) ; -r \leq u \leq 0\}$ is the segment of the process $X$, while $X(t)$ is $\mathbb{R}^d$-valued at time $t$. Since the coefficients depend on the past histories of the process, the solution process is non-Markovian, and we cannot use any fruitful techniques of partial differential equations and potential theory. In this talk, we shall study some properties on the density for the solution under the condition that the coefficients of the diffusion terms satisfy the uniformly elliptic condition, via the Malliavin calculus.