On the Solution of Marginal–Sum Equations

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Abstract

In motor car insurance, the risks of a portfolio usually are grouped according to the realizations of two (or more) risk characteristics and it is assumed that, for every group, the expected average claim amount can be represented as a product of parameters representing the realizations of the risk characteristics. This model leads, in a natural way, to the problem of marginal–sum estimation and hence to the problem of solving marginal–sum equations. Under the assumption that at least one observation is available in every group, we show that a solution of the marginal–sum equations exists, that it is radially unique, and that it can be obtained by iteration with an arbitrary initial value.

1 Introduction

In motor car insurance, the risks of a portfolio usually are grouped according to the realizations of two or more risk characteristics.

In the case of two risk characteristics, it is assumed that, for every group, the *claim number* $N_{i,k}$ and the *aggregate claim amount* $S_{i,k}$ are related by the model equations

$$E\left[\frac{S_{i,k}}{N_{i,k}}\right] \quad = \quad \mu\,\alpha_i\,\beta_k$$

where μ is a scale parameter and the parameters α_i and β_k with $i \in \{1,\ldots,I\}$ and $k \in \{1,\ldots,K\}$ refer to the realizations of the risk characteristics.

Since expectations of quotients are difficult to handle in general, it is convenient to replace expectations of quotients by quotients of expectations and hence to replace the previous model equations by the equations

$$\mu\,\alpha_i\,\beta_k\,E[N_{i,k}] \quad = \quad E[S_{i,k}]$$

Summation yields

$$\sum_{l=1}^{K} \mu\, \alpha_i\, \beta_l\, E[N_{i,l}] \;=\; \sum_{l=1}^{K} E[S_{i,l}] \qquad \text{and} \qquad \sum_{j=1}^{I} \mu\, \alpha_j\, \beta_k\, E[N_{j,k}] \;=\; \sum_{j=1}^{I} E[S_{j,k}]$$

with $i \in \{1, \ldots, I\}$ and $k \in \{1, \ldots, K\}$. Replacing all parameters and expectations in these equations by random variables, we obtain the *marginal–sum equations*

$$\sum_{l=1}^{K} \widehat{\mu}\, \widehat{\alpha}_i\, \widehat{\beta}_l\, N_{i,l} \;=\; \sum_{l=1}^{K} S_{i,l} \qquad \text{and} \qquad \sum_{j=1}^{I} \widehat{\mu}\, \widehat{\alpha}_j\, \widehat{\beta}_k\, N_{j,k} \;=\; \sum_{j=1}^{I} S_{j,k}$$

with $i \in \{1, \ldots, I\}$ and $k \in \{1, \ldots, K\}$. We say that random variables $\widehat{\mu}, \widehat{\alpha}_1, \ldots, \widehat{\alpha}_I, \widehat{\beta}_1, \ldots, \widehat{\beta}_K$ are *marginal–sum estimators* of the parameters $\mu, \alpha_1, \ldots, \alpha_I, \beta_1, \ldots, \beta_K$ if they solve the marginal–sum equations.

Although it seems to be unknown until now whether or not marginal–sum estimators exist, their existence is usually taken as granted and iteration methods are used to determine approximations. The present paper is intended as a first but essential step to close this gap.

In spite of the probabilistic model justifying marginal–sum estimation, the problem of the existence of marginal–sum estimators is essentially non–probabilistic. We thus propose a deterministic approach to the solution of the marginal equations (Section 2) and relate this problem to a fixed point problem (Section 3). Under the assumption that all claim numbers are strictly positive and that the right hand side of each of the marginal–sum equations is strictly positive as well, we show that the marginal–sum equations have a solution (Section 4) which is radially unique (in a sense to be made precise) and can be obtained by iteration (Section 5). We conclude with some remarks on impossible and possible extensions of our results (Section 6) and on marginal–sum estimation in loss reserving (Section 7).

2 The Marginal–Sum Equations

Throughout this paper, we consider two matrices $\mathbf{N}, \mathbf{S} \in \mathbb{R}_+^I \times \mathbb{R}_+^K$ having the property that every row vector and every column vector of each of $\mathbf{N}$ and $\mathbf{S}$ has at least one coordinate which is nonzero; this means that $\mathbf{e}_i'\mathbf{N}\mathbf{1}, \mathbf{1}'\mathbf{N}\mathbf{e}_k, \mathbf{e}_i'\mathbf{S}\mathbf{1}, \mathbf{1}'\mathbf{S}\mathbf{e}_k > 0$ holds for all $i \in \{1, \ldots, I\}$ and $k \in \{1, \ldots, K\}$ (where $\mathbf{e}_i \in \mathbb{R}^I$ and $\mathbf{e}_k \in \mathbb{R}^K$ are unit vectors and $\mathbf{1}$ is a vector of suitable dimension with all coordinates being equal to 1). The assumption on the matrix $\mathbf{N}$ will be strengthened in Sections 4 and 5 below.

We are interested in solutions $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in (0, \infty) \times (0, \infty)^I \times (0, \infty)^K$ of the *marginal–sum equations*

$$\sum_{l=1}^{K} \mu\, \alpha_i\, \beta_l\, N_{i,l} \;=\; \sum_{l=1}^{K} S_{i,l} \qquad \text{and} \qquad \sum_{j=1}^{I} \mu\, \alpha_j\, \beta_k\, N_{j,k} \;=\; \sum_{j=1}^{I} S_{j,k}$$

with $i \in \{1, \ldots, I\}$ and $k \in \{1, \ldots, K\}$.

The marginal–sum equations have either no solution or infinitely many solutions: Indeed, if $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a solution, then $(1, \mu\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(1, \boldsymbol{\alpha}, \mu\boldsymbol{\beta})$ are solutions as well; more generally, if $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a solution and if $c, d \neq 0$, then $((cd)^{-1}\mu, c\boldsymbol{\alpha}, d\boldsymbol{\beta})$ is a solution as well.

The parameter μ is obviously of minor interest: Indeed, if $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a solution of the marginal equations, then

$$\mu \;=\; \frac{\sum_{j=1}^{I} \sum_{l=1}^{K} S_{j,l}}{\sum_{j=1}^{I} \sum_{l=1}^{K} \alpha_j \, N_{j,l} \, \beta_l}$$

This shows that the parameter μ is nothing else than a scaling parameter which permits any scaling of the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ by a strictly positive factor. For example, the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ may be scaled by the requirement that either

$$\mathbf{1}'\boldsymbol{\alpha} \;=\; 1 \qquad \text{and} \qquad \mathbf{1}'\boldsymbol{\beta} \;=\; 1$$

or

$$\max_{j \in \{1,\dots,I\}} \alpha_j \;=\; 1 \qquad \text{and} \qquad \max_{l \in \{1,\dots,K\}} \beta_l \;=\; 1$$

Scaling the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ may be of substantial interest in actuarial practice.

Because of the preceding discussion, we now simplify the terminology by saying that the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in (0,\infty)^I \times (0,\infty)^K$ is a *solution* of the marginal–sum equations if there exists some $\mu \in (0,\infty)$ such that the triplet $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a solution of the marginal–sum equations; in that case, we have

$$\mu \;=\; \frac{\mathbf{1}'\mathbf{S}\mathbf{1}}{\boldsymbol{\alpha}'\mathbf{N}\boldsymbol{\beta}}$$

We also say that the marginal–sum equations have a *radially unique solution* if they have a solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ and if every solution $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies

$$\boldsymbol{\alpha} \;=\; c\,\boldsymbol{\alpha}^* \qquad \text{and} \qquad \boldsymbol{\beta} \;=\; d\,\boldsymbol{\beta}^*$$

for some $c, d \in (0,\infty)$.

Under the rather restrictive assumption that all coordinates of the matrix $\mathbf{N}$ are identical, it is easily seen that the marginal–sum equation have a radially unique solution and that the solutions can be given in closed form:

2.1 Example. Assume that $N_{i,k} = \nu$ holds for some $\nu \in (0,\infty)$ and for all $i \in \{1,\dots,I\}$ and all $k \in \{1,\dots,K\}$. Then the marginal–sum equations can be written as

$$\mu\nu\,\boldsymbol{\alpha} \;=\; \frac{\mathbf{S}\mathbf{1}}{\mathbf{1}'\boldsymbol{\beta}} \qquad \text{and} \qquad \mu\nu\,\boldsymbol{\beta} \;=\; \frac{\mathbf{S}'\mathbf{1}}{\mathbf{1}'\boldsymbol{\alpha}}$$

Thus, letting

$$\boldsymbol{\alpha}^* := \frac{\mathbf{S1}}{\mathbf{1}'\mathbf{S1}} \qquad \text{and} \qquad \boldsymbol{\beta}^* := \frac{\mathbf{S}'\mathbf{1}}{\mathbf{1}'\mathbf{S1}}$$

we see that $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ is a solution of the marginal equations (with $\mu^* = \mathbf{1}'\mathbf{S1}/\nu$) which satisfies

$$\mathbf{1}'\boldsymbol{\alpha}^* = 1 \qquad \text{and} \qquad \mathbf{1}'\boldsymbol{\beta}^* = 1$$

Moreover, every solution $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of the marginal equations satisfies

$$\boldsymbol{\alpha} = c\,\boldsymbol{\alpha}^* \qquad \text{and} \qquad \boldsymbol{\beta} = d\,\boldsymbol{\beta}^*$$

for some $c, d \in (0, \infty)$ such that the marginal–sum equations have a radially unique solution.

In the general case, however, the existence and the radial uniqueness of solution of the marginal–sum equations are far from being evident.

3 Equivalent Fixed Point Problems

The marginal–sum equations may be expressed in terms of nonlinear maps:

Define first $\mathbf{G} : (0, \infty)^I \to (0, \infty)^K$ and $\mathbf{H} : (0, \infty)^K \to (0, \infty)^I$ coordinatewise by letting

$$G_k(\boldsymbol{\alpha}) := \frac{\mathbf{1}'\mathbf{S}\mathbf{e}_k}{\boldsymbol{\alpha}'\mathbf{N}\mathbf{e}_k} \qquad \text{and} \qquad H_i(\boldsymbol{\beta}) := \frac{\mathbf{e}_i'\mathbf{S1}}{\mathbf{e}_i'\mathbf{N}\boldsymbol{\beta}}$$

Then a pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in (0, \infty)^I \times (0, \infty)^K$ is a solution of the marginal–sum equations if and only if there exists some $\mu \in (0, \infty)$ such that

$$\mu\,\boldsymbol{\alpha} = \mathbf{H}(\boldsymbol{\beta}) \qquad \text{and} \qquad \mu\,\boldsymbol{\beta} = \mathbf{G}(\boldsymbol{\alpha})$$

In that case, we have

$$\mu = \frac{\mathbf{1}'\mathbf{S1}}{\boldsymbol{\alpha}'\mathbf{N}\boldsymbol{\beta}}$$

The following lemma is obvious:

3.1 Lemma. *Each of the maps* $\mathbf{G}$ *and* $\mathbf{H}$ *is*
(i) *homogeneous of degree* -1,
(ii) *monotone decreasing, and*
(iii) *continuous.*

Here and in the sequel, inequalities between vectors, order intervals, and monotonicity of maps refer to the coordinatewise order relation on the Euclidean space.

Define now $\mathbf{\Phi} : (0,\infty)^I \to (0,\infty)^I$ and $\mathbf{\Psi} : (0,\infty)^K \to (0,\infty)^K$ by letting

$$\mathbf{\Phi} \;:=\; \mathbf{H} \circ \mathbf{G} \qquad \text{and} \qquad \mathbf{\Psi} \;:=\; \mathbf{G} \circ \mathbf{H}$$

The following lemma records some basic properties of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ which are evident from the definitions and Lemma 3.1:

3.2 Lemma. *Each of the maps* $\mathbf{\Phi}$ *and* $\mathbf{\Psi}$ *is*
(i) *homogeneous of degree* 1,
(ii) *monotone increasing*, *and*
(iii) *continuous.*
Moreover, *the maps* $\mathbf{\Phi}$ *and* $\mathbf{\Psi}$ *satisfy* $\mathbf{G} \circ \mathbf{\Phi} = \mathbf{\Psi} \circ \mathbf{G}$ *and* $\mathbf{\Phi} \circ \mathbf{H} = \mathbf{H} \circ \mathbf{\Psi}$.

The following result relates the solutions of the marginal–equations and the fixed points of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ to each other:

3.3 Theorem.
(1) *If* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *is a solution of the marginal equations*, *then* $\boldsymbol{\alpha}$ *is a fixed point of* $\mathbf{\Phi}$ *and* $\boldsymbol{\beta}$ *is a fixed point of* $\mathbf{\Psi}$.
(2) *If* $\boldsymbol{\alpha}$ *is a fixed point of* $\mathbf{\Phi}$, *then* $\mathbf{G}(\boldsymbol{\alpha})$ *is a fixed point of* $\mathbf{\Psi}$ *and* $(\boldsymbol{\alpha}, \mathbf{G}(\boldsymbol{\alpha}))$ *is a solution of the marginal equations.*
(3) *If* $\boldsymbol{\beta}$ *is a fixed point of* $\mathbf{\Psi}$, *then* $\mathbf{H}(\boldsymbol{\beta})$ *is a fixed point of* $\mathbf{\Phi}$ *and* $(\mathbf{H}(\boldsymbol{\beta}), \boldsymbol{\beta})$ *is a solution of the marginal equations.*

Proof. Assume first that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is a solution of the marginal equations and define

$$\mu \;:=\; \frac{\mathbf{1}'\mathbf{S}\mathbf{1}}{\boldsymbol{\alpha}'\mathbf{N}\boldsymbol{\beta}}$$

Lemma 3.1 yields

$$\begin{aligned} \mu\,\boldsymbol{\alpha} &= \mathbf{H}(\boldsymbol{\beta}) \\ &= \mathbf{H}(\mu^{-1}\,\mathbf{G}(\boldsymbol{\alpha})) \\ &= \mu\,\mathbf{H}(\mathbf{G}(\boldsymbol{\alpha})) \\ &= \mu\,\mathbf{\Phi}(\boldsymbol{\alpha}) \end{aligned}$$

and hence

$$\boldsymbol{\alpha} \;=\; \mathbf{\Phi}(\boldsymbol{\alpha})$$

By symmetry, we obtain

$$\boldsymbol{\beta} \;=\; \mathbf{\Psi}(\boldsymbol{\beta})$$

This proves (1).

Assume now that $\boldsymbol{\alpha}$ is a fixed point of $\boldsymbol{\Phi}$. Then we have

$$\boldsymbol{\Phi}(\boldsymbol{\alpha}) \;=\; \boldsymbol{\alpha}$$

and Lemma 3.2 yields

$$\begin{aligned}\boldsymbol{\Psi}(\mathbf{G}(\boldsymbol{\alpha})) &= \mathbf{G}(\boldsymbol{\Phi}(\boldsymbol{\alpha})) \\ &= \mathbf{G}(\boldsymbol{\alpha})\end{aligned}$$

which means that $\mathbf{G}(\boldsymbol{\alpha})$ is a fixed point of $\boldsymbol{\Psi}$. Furthermore, letting

$$\boldsymbol{\beta} \;:=\; \mathbf{G}(\boldsymbol{\alpha})$$

we obtain

$$\begin{aligned}\boldsymbol{\alpha} &= \boldsymbol{\Phi}(\boldsymbol{\alpha}) \\ &= \mathbf{H}(\mathbf{G}(\boldsymbol{\alpha})) \\ &= \mathbf{H}(\boldsymbol{\beta})\end{aligned}$$

which means that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is a solution of the marginal–sum equations (with $\mu = 1$). This proves (2), and (3) follows by symmetry. $\square$

We say that a map $\boldsymbol{\Gamma} : (0,\infty)^d \to (0,\infty)^d$ has a *radially unique fixed point* if $\boldsymbol{\Gamma}$ has a fixed point $\mathbf{z}^*$ and if every fixed point $\mathbf{z}$ of $\boldsymbol{\Gamma}$ satisfies

$$\mathbf{z} \;=\; c\,\mathbf{z}^*$$

for some $c \in (0,\infty)$.

3.4 Theorem. *The following are equivalent*:
(a) *The marginal–sum equations have a radially unique solution.*
(b) *The map $\boldsymbol{\Phi}$ has a radially unique fixed point.*
(c) *The map $\boldsymbol{\Psi}$ has a radially unique fixed point.*

Proof. Assume first that the marginal–sum equations have a radially unique solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ and consider any fixed point $\boldsymbol{\alpha}$ of $\boldsymbol{\Phi}$. By Theorem 3.3, $(\boldsymbol{\alpha}, \mathbf{G}(\boldsymbol{\alpha}))$ is a solution of the marginal–sum equations and it now follows that there exists some $c \in (0,\infty)$ such that

$$\boldsymbol{\alpha} \;=\; c\,\boldsymbol{\alpha}^*$$

Therefore, (a) implies (b), and it follows by symmetry that (a) implies (c).
Assume now that $\boldsymbol{\Phi}$ has a radially unique fixed point $\boldsymbol{\alpha}^*$ and consider any fixed point $\boldsymbol{\beta}$ of $\boldsymbol{\Psi}$. By Theorem 3.3,

$$\boldsymbol{\beta}^* \;:=\; \mathbf{G}(\boldsymbol{\alpha}^*)$$

is a fixed point of $\mathbf{\Psi}$ and $\mathbf{H}(\boldsymbol{\beta})$ is a fixed point of $\mathbf{\Phi}$. It now follows that there exists some $c \in (0,\infty)$ such that

$$\mathbf{H}(\boldsymbol{\beta}) \;=\; c\,\boldsymbol{\alpha}^*$$

and we obtain

$$\begin{aligned}
\boldsymbol{\beta} &= \mathbf{\Psi}(\boldsymbol{\beta}) \\
&= \mathbf{G}(\mathbf{H}(\boldsymbol{\beta})) \\
&= \mathbf{G}(c\,\boldsymbol{\alpha}^*) \\
&= c^{-1}\,\mathbf{G}(\boldsymbol{\alpha}^*) \\
&= c^{-1}\,\boldsymbol{\beta}^*
\end{aligned}$$

Therefore, (b) implies (c), and it follows by symmetry that (c) implies (b).
Assume finally that $\mathbf{\Phi}$ has a radially unique fixed point $\boldsymbol{\alpha}^*$ and consider any solution $(\boldsymbol{\alpha},\boldsymbol{\beta})$ of the marginal–sum equations. By Theorem 3.3, $\boldsymbol{\alpha}$ is a fixed point of $\mathbf{\Phi}$ and $\boldsymbol{\beta}$ is a fixed point of $\mathbf{\Psi}$. By what we have shown before, we know that also $\mathbf{\Psi}$ has a radially unique fixed point $\boldsymbol{\beta}^*$. It now follows that there exist some $c, d \in (0,\infty)$ such that

$$\boldsymbol{\alpha} \;=\; c\,\boldsymbol{\alpha}^* \qquad \text{and} \qquad \boldsymbol{\beta} \;=\; d\,\boldsymbol{\beta}^*$$

Therefore, (b) implies (a), and it follows by symmetry that (c) implies (a). $\square$

4 Existence of a Fixed Point of $\mathbf{\Phi}$

In the present section, we assume that all coordinates of $\mathbf{N}$ are strictly positive; this means that $\mathbf{e}_i'\mathbf{N}\mathbf{e}_k > 0$ holds for all $i \in \{1,\ldots,I\}$ and $k \in \{1,\ldots,K\}$

Define $\overline{\mathbf{G}} : \mathbb{R}_+^I \setminus \{\mathbf{0}\} \to (0,\infty)^K$ and $\overline{\mathbf{H}} : \mathbb{R}_+^K \setminus \{\mathbf{0}\} \to (0,\infty)^I$ coordinatewise by letting

$$\overline{G}_k(\boldsymbol{\alpha}) \;:=\; \frac{\mathbf{1}'\mathbf{S}\mathbf{e}_k}{\boldsymbol{\alpha}'\mathbf{N}\mathbf{e}_k} \qquad \text{and} \qquad \overline{H}_i(\boldsymbol{\beta}) \;:=\; \frac{\mathbf{e}_i'\mathbf{S}\mathbf{1}}{\mathbf{e}_i'\mathbf{N}\boldsymbol{\beta}}$$

(where $\mathbf{0}$ is a vector of suitable dimension with all coordinates being equal to 0) and define $\overline{\mathbf{\Phi}} : \mathbb{R}_+^I \setminus \{\mathbf{0}\} \to (0,\infty)^I$ and $\overline{\mathbf{\Psi}} : \mathbb{R}_+^K \setminus \{\mathbf{0}\} \to (0,\infty)^K$ by letting

$$\overline{\mathbf{\Phi}} \;:=\; \overline{\mathbf{H}} \circ \overline{\mathbf{G}} \qquad \text{and} \qquad \overline{\mathbf{\Psi}} \;:=\; \overline{\mathbf{G}} \circ \overline{\mathbf{H}}$$

Then $\overline{\mathbf{\Phi}}$ extends $\mathbf{\Phi}$, and $\overline{\mathbf{\Psi}}$ extends $\mathbf{\Psi}$. By symmetry, it is sufficient to study the properties of $\overline{\mathbf{\Phi}}$.

We can now prove the existence of a fixed point of $\mathbf{\Phi}$:

4.1 Theorem. *The map $\mathbf{\Phi}$ has a fixed point.*

Proof. Let

$$\Delta^I \quad := \quad \{\boldsymbol{\alpha} \in \mathbb{R}^I_+ \mid \mathbf{1}'\boldsymbol{\alpha} = 1\}$$

and define $\widetilde{\boldsymbol{\Phi}} : \Delta^I \to \Delta^I$ by letting

$$\widetilde{\boldsymbol{\Phi}}(\boldsymbol{\alpha}) \quad := \quad \frac{1}{1 + \mathbf{1}'\overline{\boldsymbol{\Phi}}(\boldsymbol{\alpha})} \Big(\boldsymbol{\alpha} + \overline{\boldsymbol{\Phi}}(\boldsymbol{\alpha})\Big)$$

Then the set Δ^I is nonempty, convex and compact, and the map $\widetilde{\boldsymbol{\Phi}}$ is continuous since $\overline{\boldsymbol{\Phi}}$ is continuous. It now follows from Brouwer's Fixed Point Theorem that $\widetilde{\boldsymbol{\Phi}}$ has a fixed point; see e. g. Granas and Dugundji [2003; Theorems 7.1 and 7.2]. Assume now that $\boldsymbol{\alpha}$ is a fixed point of $\widetilde{\boldsymbol{\Phi}}$. Then we have $\widetilde{\boldsymbol{\Phi}}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ and hence

$$\overline{\boldsymbol{\Phi}}(\boldsymbol{\alpha}) \quad = \quad \lambda\, \boldsymbol{\alpha}$$

with $\lambda := \mathbf{1}'\overline{\boldsymbol{\Phi}}(\boldsymbol{\alpha})$. For all $i \in \{1, \dots, I\}$, we thus have $\overline{\Phi}_i(\boldsymbol{\alpha}) = \lambda\, \alpha_i$ and hence

$$\sum_{l=1}^K S_{i,l} \quad = \quad \lambda\, \alpha_i \sum_{l=1}^K N_{i,l} \frac{\sum_{j=1}^I S_{j,l}}{\sum_{j=1}^I \alpha_j\, N_{j,l}}$$

Summation yields $\sum_{i=1}^I \sum_{l=1}^K S_{i,l} = \lambda \sum_{l=1}^K \sum_{j=1}^I S_{j,l}$ and hence $\lambda = 1$. We thus obtain

$$\overline{\boldsymbol{\Phi}}(\boldsymbol{\alpha}) \quad = \quad \boldsymbol{\alpha}$$

which means that $\boldsymbol{\alpha}$ is also a fixed point of $\overline{\boldsymbol{\Phi}}$. Since $\boldsymbol{\alpha} = \overline{\boldsymbol{\Phi}}(\boldsymbol{\alpha}) \in (0, \infty)^I$, we see that $\boldsymbol{\alpha}$ is even a fixed point of $\boldsymbol{\Phi}$. $\square$

It now follows from Theorems 4.1 and 3.3 that the marginal–sum equations have a solution whenever all entries of the matrix $\mathbf{N}$ are strictly positive.

5 Convergence of Iterates and Radial Uniqueness of the Fixed Points of $\boldsymbol{\Phi}$

As in the previous section, we assume that all coordinates of $\mathbf{N}$ are strictly positive and we consider only the map $\boldsymbol{\Phi}$.

5.1 Lemma. *Every coordinate of* $\boldsymbol{\Phi}$ *is partially differentiable. Moreover, for every interval* $[\mathbf{u}, \mathbf{z}] \subseteq (0, \infty)^I$, *there exists some* $\gamma \in (0, \infty)$ *such that*

$$\frac{\partial \Phi_i}{\partial t_j}(\mathbf{t}) \;\geq\; \gamma$$

holds for all $i, j \in \{1, \dots, I\}$ *and* $\mathbf{t} \in [\mathbf{u}, \mathbf{z}]$.

Proof. For $i, j \in \{1, \dots, I\}$, define $U_i, Z_{i,j} : (0,\infty)^I \to (0,\infty)$ by letting

$$U_i(\mathbf{t}) \;:=\; \frac{\displaystyle\sum_{l=1}^{K} S_{i,l}}{\left(\displaystyle\sum_{l=1}^{K} N_{i,l} \, \frac{\sum_{h=1}^{I} S_{h,l}}{\sum_{h=1}^{I} t_h \, N_{h,l}}\right)^2}$$

and

$$Z_{i,j}(\mathbf{t}) \;:=\; \sum_{l=1}^{K} N_{i,l} \, N_{j,l} \, \frac{\sum_{h=1}^{I} S_{h,l}}{\left(\sum_{h=1}^{I} t_h \, N_{h,l}\right)^2}$$

Then U_i is monotone increasing and $Z_{i,j}$ is monotone decreasing, and we obtain

$$\begin{aligned} \frac{\partial \Phi_i}{\partial t_j}(\mathbf{t}) &= U_i(\mathbf{t}) \, Z_{i,j}(\mathbf{t}) \\ &\geq U_i(\mathbf{u}) \, Z_{i,j}(\mathbf{z}) \end{aligned}$$

for all $\mathbf{t} \in [\mathbf{u}, \mathbf{z}]$. Letting $\gamma := \min_{i,j \in \{1,\dots,I\}} U_i(\mathbf{u}) \, Z_{i,j}(\mathbf{z})$ yields the assertion. $\square$

By Theorem 4.1, the map $\mathbf{\Phi}$ has a fixed point. Using Lemma 5.1, we can now prove the following result on the convergence of the iterates under $\mathbf{\Phi}$ of an arbitrary vector in the domain of $\mathbf{\Phi}$:

5.2 Theorem. *Let $\boldsymbol{\alpha}$ be a fixed point of $\mathbf{\Phi}$. Then, for every $\mathbf{x} \in (0,\infty)^I$, there exists some $\xi \in (0,\infty)$ such that the sequence $\{\mathbf{\Phi}^n(\mathbf{x})\}_{n \in \mathbb{N}_0}$ converges to $\xi \boldsymbol{\alpha}$.*

Proof. For all $n \in \mathbb{N}_0$, define

$$\mathbf{x}^{(n)} \;:=\; \mathbf{\Phi}^n(\mathbf{x})$$

as well as

$$\mathbf{u}^{(n)} \;:=\; \lambda^{(n)} \boldsymbol{\alpha} \qquad \text{and} \qquad \mathbf{z}^{(n)} \;:=\; \mu^{(n)} \boldsymbol{\alpha}$$

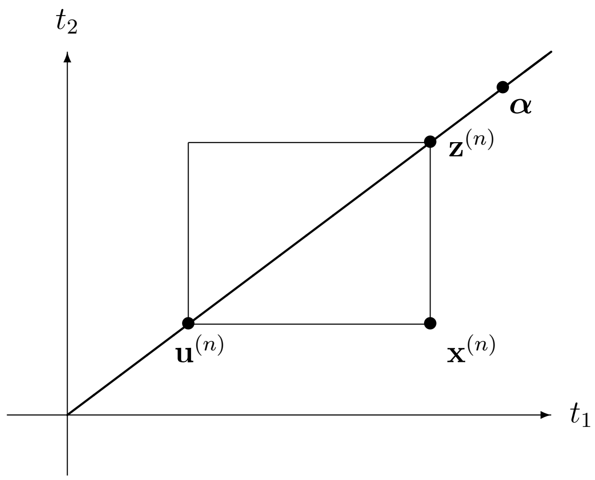
with

$$\lambda^{(n)} \;:=\; \min_{i \in \{1,\dots,I\}} \frac{x_i^{(n)}}{\alpha_i} \qquad \text{and} \qquad \mu^{(n)} \;:=\; \max_{i \in \{1,\dots,I\}} \frac{x_i^{(n)}}{\alpha_i}$$

Then we have

$$\mathbf{u}^{(n)} \;\leq\; \mathbf{x}^{(n)} \;\leq\; \mathbf{z}^{(n)}$$

This construction is illustrated by the following picture:



Since $\mathbf{u}^{(n)}$ and $\mathbf{z}^{(n)}$ are fixed points of $\boldsymbol{\Phi}$ and since $\boldsymbol{\Phi}$ is monotone increasing, we obtain $\lambda^{(n)}\boldsymbol{\alpha} = \mathbf{u}^{(n)} \leq \mathbf{x}^{(n+1)} \leq \mathbf{z}^{(n)} = \mu^{(n)}\boldsymbol{\alpha}$ and hence

$$\lambda^{(n)} \ \leq \ \lambda^{(n+1)} \ \ \leq \ \ \mu^{(n+1)} \ \leq \ \mu^{(n)}$$

Therefore, there exist some $\lambda, \mu \in (0, \infty)$ satisfying

$$\lambda \ = \ \lim_{n\to\infty} \lambda^{(n)} \qquad \text{and} \qquad \mu \ = \ \lim_{n\to\infty} \mu^{(n)}$$

as well as

$$\lambda \ \leq \ \mu$$

We shall now prove that $\lambda = \mu$.

– Assume first that $\lambda^{(m)} = \mu^{(m)}$ holds for some $m \in \mathbb{N}_0$. Then we have

$$\begin{aligned} \lambda \ &= \ \lambda^{(m)} \\ &= \ \mu^{(m)} \\ &= \ \mu \end{aligned}$$

– Assume now that $\lambda^{(n)} < \mu^{(n)}$ holds for all $n \in \mathbb{N}_0$. For all $n \in \mathbb{N}_0$, define

$$\begin{aligned} \mathbf{v}^{(n)} \ &:= \ \mathbf{u}^{(n)} + \Bigl(\mu^{(n)} - \lambda^{(n)}\Bigr)\alpha_{i(n)}\,\mathbf{e}_{i(n)} \\ \mathbf{y}^{(n)} \ &:= \ \mathbf{z}^{(n)} + \Bigl(\lambda^{(n)} - \mu^{(n)}\Bigr)\alpha_{j(n)}\,\mathbf{e}_{j(n)} \end{aligned}$$

where $i(n), j(n) \in \{1, \ldots, I\}$ satisfy

$$\frac{x^{(n)}_{j(n)}}{\alpha_{j(n)}} \ = \ \lambda^{(n)} \qquad \text{and} \qquad \frac{x^{(n)}_{i(n)}}{\alpha_{i(n)}} \ = \ \mu^{(n)}$$

Then we have

$$\mathbf{u}^{(n)} \ \leq \ \mathbf{v}^{(n)} \ \leq \ \mathbf{x}^{(n)} \ \leq \ \mathbf{y}^{(n)} \ \leq \ \mathbf{z}^{(n)}$$

Define

$$\gamma \;:=\; \min_{i,j\in\{1,\dots,I\}} \inf_{\mathbf{t}\in[\mathbf{u}^{(0)},\mathbf{z}^{(0)}]} \frac{\partial\Phi_i}{\partial t_j}(\mathbf{t})$$

By Lemma 5.1, we have $\gamma \in (0,\infty)$.
Since $\mathbf{z}^{(n)}$ is a fixed point of $\boldsymbol{\Phi}$ and since $\boldsymbol{\Phi}$ is monotone increasing, we obtain

$$\begin{aligned}\mathbf{z}^{(n)} - \mathbf{x}^{(n+1)} &= \boldsymbol{\Phi}(\mathbf{z}^{(n)}) - \boldsymbol{\Phi}(\mathbf{x}^{(n)})\\ &\geq \boldsymbol{\Phi}(\mathbf{z}^{(n)}) - \boldsymbol{\Phi}(\mathbf{y}^{(n)})\end{aligned}$$

Now, for every $i \in \{1,\dots,I\}$, the mean value theorem yields the existence of some $\mathbf{t}^{(n,i)} \in [\mathbf{y}^{(n)},\mathbf{z}^{(n)}] \subseteq [\mathbf{u}^{(0)},\mathbf{z}^{(0)}]$ such that

$$\begin{aligned}z_i^{(n)} - x_i^{(n+1)} &\geq \Phi_i(\mathbf{z}^{(n)}) - \Phi_i(\mathbf{y}^{(n)})\\ &= \left(\frac{\partial\Phi_i}{\partial t_{j(n)}}(\mathbf{t}^{(n,i)})\right)\left(z_{j(n)}^{(n)} - y_{j(n)}^{(n)}\right)\\ &= \left(\frac{\partial\Phi_i}{\partial t_{j(n)}}(\mathbf{t}^{(n,i)})\right)\left(\mu^{(n)} - \lambda^{(n)}\right)\alpha_{j(n)}\\ &\geq \gamma\left(\mu^{(n)} - \lambda^{(n)}\right)\min_{j\in\{1,\dots,I\}}\alpha_j\end{aligned}$$

Repeating the argument yields

$$x_i^{(n+1)} - u_i^{(n)} \;\geq\; \gamma\left(\mu^{(n)} - \lambda^{(n)}\right)\min_{j\in\{1,\dots,I\}}\alpha_j$$

for all $i \in \{1,\dots,I\}$. We thus obtain

$$\begin{aligned}\mathbf{z}^{(n+1)} &= \mu^{(n+1)}\,\boldsymbol{\alpha}\\ &= \frac{x_{i(n+1)}^{(n+1)}}{\alpha_{i(n+1)}}\,\boldsymbol{\alpha}\\ &\leq \frac{1}{\alpha_{i(n+1)}}\left(z_{i(n+1)}^{(n)} - \gamma\left(\mu^{(n)} - \lambda^{(n)}\right)\min_{j\in\{1,\dots,I\}}\alpha_j\right)\boldsymbol{\alpha}\\ &\leq \left(\mu^{(n)} - \gamma\left(\mu^{(n)} - \lambda^{(n)}\right)\frac{\min_{j\in\{1,\dots,I\}}\alpha_j}{\max_{j\in\{1,\dots,I\}}\alpha_j}\right)\boldsymbol{\alpha}\end{aligned}$$

as well as

$$\mathbf{u}^{(n+1)} \;\geq\; \left(\lambda^{(n)} + \gamma\left(\mu^{(n)} - \lambda^{(n)}\right)\frac{\min_{j\in\{1,\dots,I\}}\alpha_j}{\max_{j\in\{1,\dots,I\}}\alpha_j}\right)\boldsymbol{\alpha}$$

Combining these two inequalities, we obtain

$$\begin{aligned}\left(\mu^{(n+1)} - \lambda^{(n+1)}\right)\boldsymbol{\alpha} &= \mathbf{z}^{(n+1)} - \mathbf{u}^{(n+1)}\\ &\leq \left(\mu^{(n)} - \lambda^{(n)}\right)\left(1 - 2\,\gamma\,\frac{\min_{j\in\{1,\dots,I\}}\alpha_j}{\max_{j\in\{1,\dots,I\}}\alpha_j}\right)\boldsymbol{\alpha}\end{aligned}$$

and hence

$$0 \;\le\; \mu^{(n+1)} - \lambda^{(n+1)} \;\le\; \Bigl(\mu^{(n)} - \lambda^{(n)}\Bigr)\left(1 - 2\,\gamma\,\frac{\min_{j\in\{1,\ldots,I\}}\alpha_j}{\max_{j\in\{1,\ldots,I\}}\alpha_j}\right)$$

for all $n \in \mathbb{N}_0$. Since $\gamma \in (0,\infty)$, we have

$$1 - 2\,\gamma\,\frac{\min_{j\in\{1,\ldots,I\}}\alpha_j}{\max_{j\in\{1,\ldots,I\}}\alpha_j} \;<\; 1$$

and it follows that

$$\begin{aligned}\lambda &= \lim_{n\to\infty}\lambda^{(n)}\\ &= \lim_{n\to\infty}\mu^{(n)}\\ &= \mu\end{aligned}$$

Therefore, we have in both cases $\lambda = \mu$ and this yields

$$\begin{aligned}\lim_{n\to\infty}\mathbf{u}^{(n)} &= \lambda\,\boldsymbol{\alpha}\\ &= \mu\,\boldsymbol{\alpha}\\ &= \lim_{n\to\infty}\mathbf{z}^{(n)}\end{aligned}$$

Since $\mathbf{u}^{(n)} \le \mathbf{x}^{(n)} \le \mathbf{z}^{(n)}$, it follows that the sequence $\{\mathbf{x}^{(n)}\}_{n\in\mathbb{N}_0}$ converges to the fixed point $\lambda\,\boldsymbol{\alpha} = \mu\,\boldsymbol{\alpha}$ of $\boldsymbol{\Phi}$. □

As an immediate consequence of the previous result, we obtain the radial uniqueness of the fixed points of $\boldsymbol{\Phi}$:

5.3 Theorem. *The map $\boldsymbol{\Phi}$ has a radially unique fixed point.*

It now follows from Theorems 5.3 and 3.4 that the marginal–sum equations have a radially unique solution whenever all coordinates of the matrix $\mathbf{N}$ are strictly positive.

The proof of Theorem 5.2 uses a method presented in Morishima [1964; Appendix, Section 2] for the solution of a nonlinear eigenvalue problem. An alternative proof could be given by using Krasnoselskii [1964; Theorems 5.5 and 5.6]. The radial uniqueness of fixed points, which in present paper is obtained as a consequence of Theorem 5.2, could also be obtained from results in Morishima [1964; Appendix, Section 2].

6 Possible Extensions

Since the data represented by the matrices $\mathbf{N}$ and $\mathbf{S}$ are realizations of random matrices, it may happen that at least one class $(i,k) \in \{1,\ldots,I\}\times\{1,\ldots,K\}$ contains no observations such that the number of claims $N_{i,k}$ and hence also the aggregate claim size $S_{i,k}$ is equal to zero.

The following example shows that missing observations may cause a serious problem:

6.1 Example. Let

$$\mathbf{N} := \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{S} := \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

Then the marginal–sum equations become

$$\begin{aligned} 2\mu\alpha_1\beta_1 &= 1 \\ 3\mu\alpha_2\beta_1 + \ \mu\alpha_2\beta_2 &= 4 \\ 2\mu\beta_1\alpha_1 + 3\mu\beta_1\alpha_2 &= 1 \\ \mu\beta_2\alpha_2 &= 4 \end{aligned}$$

and the map $\mathbf{\Phi} : (0,\infty)^2 \to (0,\infty)^2$ satisfies

$$\mathbf{\Phi}(\boldsymbol{\alpha}) = \begin{pmatrix} \dfrac{1}{2/(2\alpha_1+3\alpha_2)} \\[2ex] \dfrac{4}{3/(2\alpha_1+3\alpha_2) + 4/\alpha_2} \end{pmatrix}$$

Iteration with the initial value $\mathbf{x} := (1/2)\,\mathbf{1}$ yields the following scaled values of the iterates:

n	$x_1^{(n)}$	$x_2^{(n)}$
0	0.5	0.5
1	0.742	0.258
10	0.952	0.048
100	0.995	0.005
1000	0.999	0.001
10000	1.000	0.000

One is thus tempted to conclude that $\mathbf{\Phi}$ has the fixed point $\mathbf{e}_1$ and that the pair $(\mathbf{e}_1, \mathbf{\Phi}(\mathbf{e}_1))$ is a solution of the marginal–sum equations. Unfortunately, however, $\mathbf{\Phi}(\mathbf{e}_1)$ is not defined; moreover, and this is even worse, the marginal–sum equations do not have a solution at all.
(In view of Theorem 3.3, the fact that the marginal–sum equations have no solution is in accordance with the fact that the map $\mathbf{\Phi}$ has no fixed point. Also, convergence of the iterates of $\mathbf{x} \in \mathbb{R}^2_+$ to $\mathbf{e}_1 \in \mathbb{R}^2_+$ is in accordance with the fact that the map $\mathbf{\Phi}$ has a continuous extension $\mathbb{R}^2_+ \to \mathbb{R}^2_+$ which in turn has the radially unique fixed point $\mathbf{e}_1$.)

The previous example shows that thoughtless iteration may lead to useless results.

Because of the previous example, the assumption that all coordinates of the matrix $\mathbf{N}$ are strictly positive cannot be dropped in the results of Sections 4 and 5, but it remains an open question whether or not this assumption can be relaxed.

The marginal–sum equations may also be extended to more than two risk characteristics. For example, in the case of three risk characteristics the marginal–sum equations become

$$\begin{aligned}
\sum_{l=1}^{K}\sum_{n=1}^{M} \mu\, \alpha_i\, \beta_l\, \gamma_n\, N_{i,l,n} &= \sum_{l=1}^{K}\sum_{n=1}^{M} S_{i,l,n} \\
\sum_{j=1}^{I}\sum_{n=1}^{M} \mu\, \alpha_j\, \beta_k\, \gamma_n\, N_{j,k,n} &= \sum_{j=1}^{I}\sum_{n=1}^{M} S_{j,k,n} \\
\sum_{j=1}^{I}\sum_{l=1}^{K} \mu\, \alpha_j\, \beta_l\, \gamma_m\, N_{j,l,m} &= \sum_{j=1}^{I}\sum_{l=1}^{K} S_{j,l,m}
\end{aligned}$$

with $i \in \{1, \dots, I\}$, $k \in \{1, \dots, K\}$ and $m \in \{1, \dots, M\}$. In the case of three or more risk characteristics, it is an open question whether the marginal–sum equations have a solution, whether the solution is radially unique, and whether it can be obtained by iteration.

7 Remark

Marginal–sum equations also occur in loss reserving based on run–off triangles. In that case, the data are represented in a triangle instead of a rectangle and it is well–known that the marginal–sum equations have a unique solution which justifies the chain–ladder method; see Mack [1991, 2003], Schmidt and Wünsche [1998], and Radtke and Schmidt [2004].

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