A note on Poisson renewal processes

Klaus Th. Hess and Klaus D. Schmidt

Lehrstuhl für Versicherungsmathematik
Technische Universität Dresden

Abstract

The present paper provides a short and elementary proof of the fact that the counting process generated by a renewal process with independent and identically exponentially distributed waiting times is a homogeneous Poisson process.

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{W_j\}_{j\in\mathbf{N}}$ be a sequence of random variables which are strictly positive and i. i. d. For all $n \in \mathbf{N}_0$, let

$$T_n \;:=\; \sum_{j=1}^{n} W_j$$

Then we have $T_0 = 0$ and the sequence $\{T_n\}_{n\in\mathbf{N}_0}$ is strictly increasing. For all $t \in \mathbf{R}_+$, let

$$N_t \;:=\; \sum_{n=1}^{\infty} \chi_{(0,t]} \circ T_n$$

Then we have $N_0 = 0$ and the paths of $\{N_t\}_{t\in\mathbf{R}_+}$ are right–continuous with jumps of size one and increase to infinity.

Recall that the exponential distribution with parameter $\alpha \in (0,\infty)$ is defined to be the distribution

$$\mathbf{Exp}(\alpha) \;:=\; \int \alpha e^{-\alpha w}\, \chi_{(0,\infty)}(w)\, d\boldsymbol{\lambda}(w)$$

where $\boldsymbol{\lambda}$ denotes the Lebesgue measure, and that $\{N_t\}_{t\in\mathbf{R}_+}$ is a homogeneous Poisson process with parameter $\alpha \in (0,\infty)$ if $\{N_t\}_{t\in\mathbf{R}_+}$ has independent and stationary increments with

$$P[\{N_t = k\}] \;=\; e^{-\alpha t}\,\frac{(\alpha t)^k}{k!}$$

for all $t \in (0,\infty)$ and $k \in \mathbf{N}_0$.

The following result is well–known; see e. g. Billingsley [1995; Theorem 23.1], Iranpour and Chacon [1988; Section 3.4], and Schmidt [1996; Theorem 2.3.4]:

Theorem. *Assume that the sequence* $\{W_n\}_{n\in\mathbf{N}}$ *is i. i. d. with* $P_{W_n} = \mathbf{Exp}(\alpha)$ *. Then* $\{N_t\}_{t\in\mathbf{R}_+}$ *is a homogeneous Poisson process with parameter* α *.*

The following proof of the Theorem is straightforward and completes the arguments used by Iranpour and Chacon [1988]; in particular, it is much simpler than the proof in Billingsley [1995] and the similar one in Schmidt [1996].

Proof. Consider $m \in \mathbf{N}$ and $t_0, t_1, \ldots, t_m \in \mathbf{R}_+$ such that $0 = t_0 < t_1 < \cdots < t_m$ as well as $k_1, \ldots, k_m \in \mathbf{N}_0$. For all $i \in \{0, 1, \ldots, m\}$, define

$$n_i \;:=\; \sum_{j=1}^{i} k_j$$

Then we have

$$\begin{aligned}
&P\left[\bigcap_{i=1}^{m}\{N_{t_i}-N_{t_{i-1}} = k_i\}\right] \\
&\quad= P\left[\bigcap_{i=1}^{m}\{N_{t_i} = n_i\}\right] \\
&\quad= P\left[\bigcap_{i=1}^{m}\{T_{n_i} \le t_i < T_{n_i+1}\}\right] \\
&\quad= P\left[\bigcap_{i=1}^{m}\left\{\sum_{j=1}^{n_i} W_j \le t_i < \sum_{j=1}^{n_i+1} W_j\right\}\right] \\
&\quad= \int_{\mathbf{R}^{n_m+1}} \prod_{i=1}^{m} \chi_{[0,t_i]}\left(\sum_{j=1}^{n_i} w_j\right) \chi_{(t_i,\infty)}\left(\sum_{j=1}^{n_i+1} w_j\right) \\
&\qquad\quad dP_{W_1,\ldots,W_{n_m},W_{n_m+1}}(w_1, \ldots, w_{n_m}, w_{n_m+1}) \\
&\quad= \int_{\mathbf{R}}\int_{\mathbf{R}} \cdots \int_{\mathbf{R}} \prod_{i=1}^{m} \chi_{[0,t_i]}\left(\sum_{j=1}^{n_i} w_j\right) \chi_{(t_i,\infty)}\left(\sum_{j=1}^{n_i+1} w_j\right) \\
&\qquad\quad dP_{W_1}(w_1) \cdots dP_{W_{n_m}}(w_{n_m})\, dP_{W_{n_m+1}}(w_{n_m+1}) \\
&\quad= \int_{\mathbf{R}}\int_{\mathbf{R}} \cdots \int_{\mathbf{R}} \prod_{i=1}^{m} \chi_{[0,t_i]}\left(\sum_{j=1}^{n_i} w_j\right) \chi_{(t_i,\infty)}\left(\sum_{j=1}^{n_i+1} w_j\right) \\
&\qquad\quad \cdot \prod_{j=1}^{n_m+1} \chi_{(0,\infty)}(w_j) \cdot\, \alpha^{n_m+1} e^{-\alpha \sum_{j=1}^{n_m+1} w_j}\, d\boldsymbol{\lambda}(w_1) \cdots d\boldsymbol{\lambda}(w_{n_m})\, d\boldsymbol{\lambda}(w_{n_m+1}) \\
&\quad= \int_{\mathbf{R}}\int_{\mathbf{R}} \cdots \int_{\mathbf{R}} \prod_{i=1}^{m} \chi_{[0,t_i]}(v_{n_i})\, \chi_{(t_i,\infty)}(v_{n_i+1}) \cdot \prod_{j=1}^{n_m+1} \chi_{(0,\infty)}(v_j - v_{j-1}) \\
&\qquad\quad \cdot \alpha^{n_m+1} e^{-\alpha v_{n_m+1}}\, d\boldsymbol{\lambda}(v_1) \cdots d\boldsymbol{\lambda}(v_{n_m})\, d\boldsymbol{\lambda}(v_{n_m+1})
\end{aligned}$$

with

$$v_n \;:=\; \sum_{j=1}^{n} w_j$$

for all $n \in \{0, 1, \ldots, n_m, n_m{+}1\}$. To proceed further, we partition the set $\{1, \ldots, m\}$ into

$$\begin{aligned} I(0) &:= \{i \in \{1,\ldots,m\} \mid k_i = 0\} \\ I(1) &:= \{i \in \{1,\ldots,m\} \mid k_i = 1\} \\ I(2) &:= \{i \in \{1,\ldots,m\} \mid k_i \geq 2\} \end{aligned}$$

Then we have

$$\begin{aligned}
&\prod_{i=1}^{m} \chi_{[0,t_i]}(v_{n_i})\, \chi_{(t_i,\infty)}(v_{n_i+1}) \cdot \prod_{j=1}^{n_m+1} \chi_{(0,\infty)}(v_j - v_{j-1}) \\
&\quad= \prod_{i=1}^{m} \chi_{[0,t_i]}(v_{n_i})\, \chi_{(t_i,\infty)}(v_{n_i+1}) \cdot \prod_{j=1}^{n_m} \chi_{(0,v_{j+1})}(v_j) \\
&\quad= \prod_{i=1}^{m} \chi_{[0,t_i]}(v_{n_i})\, \chi_{(t_{i-1},\infty)}(v_{n_{i-1}+1}) \cdot \prod_{j=1}^{n_m} \chi_{(0,v_{j+1})}(v_j) \cdot \chi_{(t_m,\infty)}(v_{n_m+1}) \\
&\quad= \prod_{i\in I(0)} \chi_{[0,t_i]}(v_{n_i})\, \chi_{(t_{i-1},\infty)}(v_{n_{i-1}+1}) \cdot \prod_{i\in I(1)} \chi_{[0,t_i]}(v_{n_i})\, \chi_{(t_{i-1},\infty)}(v_{n_{i-1}+1}) \\
&\qquad\quad \cdot \prod_{i\in I(2)} \chi_{[0,t_i]}(v_{n_i})\, \chi_{(t_{i-1},\infty)}(v_{n_{i-1}+1}) \cdot \prod_{j=1}^{n_m} \chi_{[0,v_{j+1})}(v_j) \cdot \chi_{(t_m,\infty)}(v_{n_m+1}) \\
&\quad= \prod_{i\in I(1)} \chi_{(t_{i-1},t_i]}(v_{n_i}) \\
&\qquad\quad \cdot \prod_{i\in I(2)} \left(\chi_{(t_{i-1},v_{n_{i-1}+2})}(v_{n_{i-1}+1}) \cdot \prod_{j=n_{i-1}+2}^{n_i-1} \chi_{(t_{i-1},v_{j+1})}(v_j) \cdot \chi_{(t_{i-1},t_i]}(v_{n_i}) \right) \\
&\qquad\quad \cdot \chi_{(t_m,\infty)}(v_{n_m+1})
\end{aligned}$$

In the last identity, we have omitted the product over $i \in I(0)$ since for all $i \in I(0)$ we have $n_i = n_{i-1}$ and hence $\chi_{[0,t_i]}(v_{n_i}) = \chi_{[0,t_i]}(v_{n_{i-1}}) \geq \chi_{[0,t_{i-1}]}(v_{n_{i-1}})$ as well as $\chi_{(t_{i-1},\infty)}(v_{n_{i-1}+1}) = \chi_{(t_{i-1},\infty)}(v_{n_i+1}) \geq \chi_{(t_i,\infty)}(v_{n_i+1})$; also, for all $i \in I(1)$ we have used the identity $n_i = n_{i-1}{+}1$. We thus obtain

$$\begin{aligned}
&\int_{\mathbf{R}} \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} \prod_{i=1}^{m} \chi_{[0,t_i]}(v_{n_i})\, \chi_{(t_i,\infty)}(v_{n_i+1}) \cdot \prod_{j=1}^{n_m+1} \chi_{(0,\infty)}(v_j - v_{j-1}) \\
&\qquad\quad \cdot \alpha^{n_m+1} e^{-\alpha v_{n_m+1}}\, d\boldsymbol{\lambda}(v_1) \cdots d\boldsymbol{\lambda}(v_{n_m})\, d\boldsymbol{\lambda}(v_{n_m+1}) \\
&\quad= \int_{\mathbf{R}} \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} \prod_{i\in I(1)} \chi_{(t_{i-1},t_i]}(v_{n_i}) \\
&\qquad\quad \cdot \prod_{i\in I(2)} \left(\chi_{(t_{i-1},v_{n_{i-1}+2})}(v_{n_{i-1}+1}) \cdot \prod_{j=n_{i-1}+2}^{n_i-1} \chi_{(t_{i-1},v_{j+1})}(v_j) \cdot \chi_{(t_{i-1},t_i]}(v_{n_i}) \right) \\
&\qquad\quad \cdot \chi_{(t_m,\infty)}(v_{n_m+1})\, \alpha^{n_m+1} e^{-\alpha v_{n_m+1}}\, d\boldsymbol{\lambda}(v_1) \cdots d\boldsymbol{\lambda}(v_{n_m})\, d\boldsymbol{\lambda}(v_{n_m+1})
\end{aligned}$$

$$
\begin{aligned}
= & \prod_{i \in I(1)} \int_{\mathbf{R}} \chi_{(t_{i-1},t_i]}(v_{n_i})\, d\boldsymbol{\lambda}(v_{n_i}) \\
& \cdot \prod_{i \in I(2)} \left(\int_{\mathbf{R}} \cdots \int_{\mathbf{R}} \chi_{(t_{i-1},v_{n_{i-1}+2})}(v_{n_{i-1}+1}) \prod_{j=n_{i-1}+2}^{n_i-1} \chi_{(t_{i-1},v_{j+1})}(v_j)\, \chi_{(t_{i-1},t_i]}(v_{n_i}) \right. \\
& \qquad\qquad \left. d\boldsymbol{\lambda}(v_{n_{i-1}+1}) \cdots d\boldsymbol{\lambda}(v_{n_i}) \right) \\
& \cdot \int_{\mathbf{R}} \chi_{(t_m,\infty)}(v_{n_m+1})\, \alpha^{n_m+1} e^{-\alpha v_{n_m+1}}\, d\boldsymbol{\lambda}(v_{n_m+1}) \\
= & \prod_{i \in I(1)} (t_i - t_{i-1}) \cdot \prod_{i \in I(2)} \frac{(t_i - t_{i-1})^{n_i - n_{i-1}}}{(n_i - n_{i-1})!} \cdot \alpha^{n_m} e^{-\alpha t_m} \\
= & \prod_{i=1}^{m} e^{-\alpha(t_i - t_{i-1})} \frac{(\alpha(t_i - t_{i-1}))^{k_i}}{k_i!}
\end{aligned}
$$

Therefore, we have

$$
P\left[\bigcap_{i=1}^{m} \{N_{t_i} - N_{t_{i-1}} = k_i\} \right] = \prod_{i=1}^{m} e^{-\alpha(t_i - t_{i-1})} \frac{(\alpha(t_i - t_{i-1}))^{k_i}}{k_i!}
$$

The assertion follows. □

Acknowledgement. When writing this paper, we were unaware of the monograph by Iranpour and Chacon [1988]. We are greatly indebted to Lothar Partzsch for providing this reference.

References

Billingsley, P. [1995]: *Probability and Measure.* Third Edition. New York – Chichester: Wiley.

Iranpour, R., and Chacon, P. [1988]: *Basic Stochastic Processes: The Mark Kac Lectures.* New York: MacMillan.

Schmidt, K. D. [1996]: *Lectures on Risk Theory.* Stuttgart: Teubner.

Klaus Th. Hess and Klaus D. Schmidt
Lehrstuhl für Versicherungsmathematik
Technische Universität Dresden
D–01062 Dresden
Germany

25th November 2003