Finite Degree Clones Are Undecidable

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A clone is a set of finitary operations closed under
- composition,
- variable identification,
- variable permutation,
- introduction of extraneous variables.

Emil Post in 1941 famously classified all Boolean clones.

Over ($\geq 3$)-element domains structure is quite complicated.
Clones are infinite. How can they be an input to an algorithm?

A clone on finite domain $A$ can be **finitely specified** in essentially 2 ways.

**First way:** Given $\mathcal{F}$, a finite set of operations of $A$, define $\text{Clo}(\mathcal{F}) = \text{“the smallest clone containing } \mathcal{F} \text{”}$.

- A with $\mathcal{F}$ forms an algebra, $\mathbb{A} = \langle A; \mathcal{F} \rangle$. Define $\text{Clo}(\mathbb{A}) = \text{Clo}(\mathcal{F})$.
- A relation of $\mathbb{A}$ is a subpower $R \subseteq A^n$ closed under $\mathcal{F}$ (hence $\text{Clo}(\mathcal{F})$)
- Define $\text{Rel}_n(\mathbb{A}) = \text{Rel}_n(\mathcal{F}) = \text{“all } (\leq n)\text{-ary relations of } \mathbb{A} \text{”}$.
- Define $\text{Rel}(\mathbb{A}) = \text{Rel}_n(\mathcal{F}) = \bigcup_{n<\infty} \text{Rel}_n(\mathbb{A})$

These are the **finitely generated** clones.

**Second way:** Given $\mathcal{R}$, a finite set of subpowers of $A$, define $\text{Pol}(\mathcal{R}) = \text{“the set of all operations of } A \text{ preserving all subpowers in } \mathcal{R} \text{”}$.

These are the **finitely related/finite degree** clones.
Rel(\mathcal{F}) = \{ R \subseteq A^n \mid R \text{ is preserved by all operations in } \mathcal{F} \}

\text{Pol}(\mathcal{R}) = \{ f : A^n \to A \mid f \text{ preserves all subpowers in } \mathcal{R} \}

These two operators form a \textbf{Galois connection}.

\begin{align*}
\mathcal{R} \subseteq \text{Rel}(\mathcal{F}) \\
\iff \mathcal{F} \subseteq \text{Pol}(\mathcal{R})
\end{align*}

Every Galois connection defines two closure operators. Here, they are

\[ \text{Clo} = \text{Pol} \circ \text{Rel} \quad \text{and} \quad \text{RClo} = \text{Rel} \circ \text{Pol}. \]

If \( \mathcal{R} \in \text{RClo}(\mathcal{S}) \), then we say "\( \mathcal{S} \) entails \( \mathcal{R} \)" and write \( \mathcal{S} \models \mathcal{R} \).

If \( f \in \text{Pol}(\mathcal{S}) \), then we say "\( \mathcal{S} \) entails \( f \)" and write \( \mathcal{S} \models f \).
For a set of relations \( S \), define

\[
\text{deg}(S) = \sup \{ \text{arity}(R) \mid R \in S \}.
\]

For a clone \( C \), define

\[
\text{deg}(C) = \inf \{ \text{deg}(S) \mid \text{Pol}(S) = C \}.
\]

For an algebra \( A \), define

\[
\text{deg}(A) = \text{deg}(\text{Clo}(A)).
\]

**The Finite Degree Problem**

Input: finite algebra \( A = \langle A; f_1, \ldots, f_n \rangle \) generating clone \( C \)

Output: whether \( \text{deg}(C) < \infty \)

(seems to originate in the 70s with the study of lattices of clones over domains of more than 2 elements)
The Finite Degree Problem

Input: finite algebra $\mathbb{A} = \langle A; f_1, \ldots, f_n \rangle$ generating clone $C$

Output: whether $\text{deg}(C) < \infty$

Given a Minsky machine $M$, we encode it into a finite algebra $\mathbb{A}(M)$.

Theorem

*The following are equivalent.*

- $M$ halts,
- $\text{deg}(\mathbb{A}(M)) < \infty$ (i.e. $\mathbb{A}(M)$ is finitely related),

Similar approaches have proved the following are undecidable:

- finite residual bound (McKenzie)
- finite axiomatizability/Tarski’s problem (McKenzie)
- certain omitting types (McKenzie, Wood)
- existence of a term op. that is NU on all but 2 elements (Maroti)
- DPSC, leading to another solution to Tarski’s problem (M)
- profiniteness (Nurakunov and Stronkowski)
Clones and The Finite Degree Problem

The Encoding of Computation

Non-halting Implies Infinite Degree

Halting Implies Finite Degree

Conclusion and Open Problems
A Minsky machine has

- registers $A$ and $B$ that have integer values $\geq 0$,

- instructions to add 1 to a register,

- instructions to test if a register is 0 and otherwise subtract 1 from it.

We can represent a Minsky machine as a finite flow graph.
How to represent intermediate computations?

- Assign a state to each node.
- A configuration \((i, \alpha, \beta)\) represents each stage of computation.
- Consider \(\mathcal{M}\) as a function, and write
  \[
  \mathcal{M}(i, \alpha, \beta) = (j, \alpha', \beta') \quad \text{or} \quad \mathcal{M}^n(i, \alpha, \beta) = (j, \alpha', \beta')
  \]
  (single step of computation or multiple).
- On \((\alpha, \beta)\), \(\mathcal{M}\) halts with registers \((1, 0)\) if \(\alpha \leq \beta\) and \((0, 1)\) otherwise.
The encoding of computation

- let $\mathbb{A}(\mathcal{M})$ be the algebra we intend to build
- configurations $(i, \alpha, \beta) \leftrightarrow$ special elements of $A(\mathcal{M})^n$
- term operations should simulate the action of $\mathcal{M}$ (need placemaker, •)
- computation on configurations $\leftrightarrow$ subalgebra generation

$\mathbb{A}(\mathcal{M})$ has universe... $A(\mathcal{M}) = \{ \langle i, c \rangle \mid i \text{ a state of } \mathcal{M}, \ c \in \{A, B, 0, \bullet, \times\} \}$

Given configuration $(k, \alpha, \beta)$ and $n \in \mathbb{N}$ define a subset of $\mathbb{A}(\mathcal{M})^n$,

$$\text{conf}(k, \alpha, \beta) = \bigcup_{p \in P_n} \left\{ p\left(\underbrace{\langle k, A \rangle, \ldots, \langle k, A \rangle}_\alpha, \underbrace{\langle k, B \rangle, \ldots, \langle k, B \rangle}_\beta, \langle k, 0 \rangle, \ldots, \langle k, 0 \rangle, \langle k, \bullet \rangle\right) \right\}_{n-\alpha-\beta-1}.$$
The encoding of computation

- term operations should simulate the action of $M$
- computation on configurations $\iff$ subalgebra generation

Design considerations

- $M(r, s) = t$ if and only if...
  - $r, s \in \text{conf}(i, \alpha, \beta)$
  - $r \neq s$
  - $t \in \text{conf}(M(i, \alpha, \beta))$
  - via some $R_+$ or $R_-$

- otherwise introduce $\times$ into the output $t$

Term operations

- $M(x, y)$ for $R_+$ or $R_-$
- $M'(x)$ for $R_-$ $\rightarrow$

- $M'(r) = t$ if and only if...
  - $r \in \text{conf}(i, \alpha, \beta)$
  - $t \in \text{conf}(M(i, \alpha, \beta))$
  - via some $R_-$ $\rightarrow$

- otherwise introduce $\times$ into the output $t$
Can we actually define $M$ and $M'$ with these features?

\[
M(x, y) = \begin{cases}
\langle j, R \rangle & \text{if } x = \langle i, \bullet \rangle, \ y = \langle i, 0 \rangle, \quad i : R^+ \rightarrow j : *, \\
\langle j, 0 \rangle & \text{if } x = \langle i, \bullet \rangle, \ y = \langle i, R \rangle, \quad i : R^- \rightarrow j : *, \\
\langle j, \bullet \rangle & \text{if } x = \langle i, 0 \rangle, \ y = \langle i, \bullet \rangle, \quad i : R^+ \rightarrow j : *, \\
\langle j, \bullet \rangle & \text{if } x = \langle i, R \rangle, \ y = \langle i, \bullet \rangle, \quad i : R^- \rightarrow j : *, \\
\langle j, c \rangle & \text{if } x = y = \langle i, c \rangle, \ c \neq \bullet, \quad i : R^+ \rightarrow j : * \quad \text{or} \quad i : R^- \rightarrow j : *, \\
\langle j, \times \rangle & \text{else if } x = \langle i, c \rangle, \ y = \langle i, d \rangle, \quad i : R^+ \rightarrow j : * \quad \text{or} \quad i : R^- \rightarrow j : *, \\
\langle i, \times \rangle & \text{otherwise, where } y = \langle i, c \rangle.
\end{cases}
\]

\[
M'(x) = \begin{cases}
\langle k, c \rangle & \text{if } x = \langle i, c \rangle, \quad i : R^+ \xrightarrow{0} k : *, \ c \neq R, \\
\langle k, \times \rangle & \text{else if } x = \langle i, R \rangle, \quad i : R^+ \xrightarrow{0} k : *, \\
\langle i, \times \rangle & \text{otherwise, where } x = \langle i, c \rangle.
\end{cases}
\]

Let’s see an example computation...
\[
\begin{align*}
1: & \quad M \begin{pmatrix}
\langle 1, \bullet \rangle, \langle 1, A \rangle \\
\langle 1, A \rangle, \langle 1, \bullet \rangle \\
\langle 1, A \rangle, \langle 1, A \rangle \\
\langle 1, B \rangle, \langle 1, B \rangle
\end{pmatrix} = \begin{pmatrix}
\langle 4, 0 \rangle \\
\langle 4, \bullet \rangle \\
\langle 4, A \rangle \\
\langle 4, B \rangle
\end{pmatrix} \\
2: & \quad M \begin{pmatrix}
\langle 4, 0 \rangle, \langle 4, 0 \rangle \\
\langle 4, \bullet \rangle, \langle 4, B \rangle \\
\langle 4, A \rangle, \langle 4, A \rangle \\
\langle 4, B \rangle, \langle 4, \bullet \rangle
\end{pmatrix} = \begin{pmatrix}
\langle 1, 0 \rangle \\
\langle 1, 0 \rangle \\
\langle 1, A \rangle \\
\langle 1, \bullet \rangle
\end{pmatrix} \\
3: & \quad M \begin{pmatrix}
\langle 1, 0 \rangle, \langle 1, 0 \rangle \\
\langle 1, 0 \rangle, \langle 1, 0 \rangle \\
\langle 1, A \rangle, \langle 1, \bullet \rangle \\
\langle 1, \bullet \rangle, \langle 1, A \rangle
\end{pmatrix} = \begin{pmatrix}
\langle 4, 0 \rangle \\
\langle 4, 0 \rangle \\
\langle 4, \bullet \rangle \\
\langle 4, 0 \rangle
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
4: & \quad M' \begin{pmatrix}
\langle 4, 0 \rangle \\
\langle 4, 0 \rangle
\end{pmatrix} = \begin{pmatrix}
\langle 5, 0 \rangle \\
\langle 5, 0 \rangle
\end{pmatrix} \\
5: & \quad M' \begin{pmatrix}
\langle 5, 0 \rangle \\
\langle 5, 0 \rangle \\
\langle 5, \bullet \rangle \\
\langle 5, 0 \rangle
\end{pmatrix} = \begin{pmatrix}
\langle 6, 0 \rangle \\
\langle 6, 0 \rangle \\
\langle 6, \bullet \rangle \\
\langle 6, 0 \rangle
\end{pmatrix} \\
6: & \quad M \begin{pmatrix}
\langle 6, 0 \rangle, \langle 6, 0 \rangle \\
\langle 6, 0 \rangle, \langle 6, 0 \rangle \\
\langle 6, \bullet \rangle, \langle 6, 0 \rangle \\
\langle 6, 0 \rangle, \langle 6, 0 \rangle
\end{pmatrix} = \begin{pmatrix}
\langle 0, 0 \rangle \\
\langle 0, 0 \rangle \\
\langle 0, B \rangle \\
\langle 0, 0 \rangle
\end{pmatrix}.
\]

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Takeaways on a relation \( R \leq A(\mathcal{M})^n \) ...

- certain elements of \( R \) encode configurations of \( \mathcal{M} \),
- \( \mathcal{M} \) and \( \mathcal{M}' \) encode the action of \( \mathcal{M} \) in the presence of these elements.

\[
\text{conf}(k, \alpha, \beta) = \bigcup_{p \in P_n} \left\{ p\left( \langle k, A \rangle, \ldots, \langle k, A \rangle, \langle k, B \rangle, \ldots, \langle k, B \rangle, \langle k, 0 \rangle, \ldots, \langle k, 0 \rangle, \langle k, \bullet \rangle \right) \right\}
\]

Questions

- What if \( R \) doesn’t contain these kinds of elements?
- What if \( R \) contains elements that aren’t “computational”? (multiple \( \bullet \)’s or non-constant states)

Call \( R \) **computational** if it doesn’t contain any elements with 2 \( \bullet \)’s or non-constant state.

The **capacity** of a computation \( \mathcal{M}^k(i, \alpha, \beta) = (j, \alpha', \beta') \) is the max sum of the registers.

The **capacity** of computational \( R \) is (number of coordinates with \( \bullet \) )−1.
We consider the halting problem on \textbf{0 register input}: $\text{config} = (1, 0, 0)$. Let $S_m = Sg_{\mathcal{A}(\mathcal{M})^m}(\text{conf}(1, 0, 0))$.

\begin{description}
\item[Theorem (The Coding Theorem)] The following are equivalent.
\begin{itemize}
  \item $\mathcal{M}^n(1, 0, 0) = (k, \alpha, \beta)$ with capacity $< m$,
  \item $\text{conf}(k, \alpha, \beta) \subseteq S_m$.
\end{itemize}
\end{description}

\begin{description}
\item[Corollary] The following are equivalent.
\begin{itemize}
  \item $\mathcal{M}$ halts with capacity $< m$,
  \item $S_m$ is halting (i.e. contains $\text{conf}(0, \alpha, \beta)$),
  \item every computational $\mathcal{R} \leq \mathcal{A}(\mathcal{M})^\ell$ with capacity $\geq m$ is halting.
\end{itemize}
\end{description}
Theorem (The Coding Theorem)

The following are equivalent.

• \( M^n(1, 0, 0) = (k, \alpha, \beta) \) with capacity \(< m\),

• \( \text{conf}(k, \alpha, \beta) \subseteq S_m \).

Framework for proving the hardness of algebraic properties

• Start out with \( \mathbb{A}(M) = \langle A(M) \ ; \ M, M' \rangle \).

• Add operations so that the property is recognizable in \( \text{Rel}(\mathbb{A}(M)) \)
  (ideally in the \( (S_m)_{m \in \mathbb{N}} \)).

• Use a computer to verify necessary computations.

• Use software development techniques:
  write unit tests, rapidly iterate the operation definitions.

This allows us to give a more unified construction for the previously
mentioned undecidability results in Universal Algebra.
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Observe

\[
\text{deg}(C) = \infty \quad \text{if and only if} \quad \forall n \ Rel_n(C) \not\models Rel(C)
\]

\[
\text{if and only if} \quad \forall n \ \exists R \ Rel_n(C) \not\models R
\]

**Idea:** to show that \(\text{deg}(\mathbb{A}(\mathcal{M})) = \infty\) when \(\mathcal{M}\) does not halt, we show the last equivalence holds for \(C = \text{Clo}(\mathbb{A}(\mathcal{M}))\).

**Two operations involved**

- semilattice operation \(\land\)
  - locally flat: \(a \land b \neq \langle \ast, \times \rangle\) iff \(a = b\)

- “initialization” operation \(I(x, y)\)
  - returns any configuration to \(\text{conf}(1, 0, 0)\)

At this point \(\mathbb{A}(\mathcal{M}) = \langle A(\mathcal{M}) ; M, M', \land, I \rangle\).
Rel\(_n(C) \models R\) if and only if \(R\) can be built from Rel\(_n(C)\) using
- intersection of equal arity relations,
- (cartesian) product of finitely many relations,
- permutation of the coordinates of a relation, and
- projection of a relation onto a subset of coordinates.

\textbf{Theorem (Zadori 1995)}

\(\text{Rel}_n(A) \models S\) if and only if

\[ S = \pi \left( \bigcap_{i \in I} \mu_i \left( \prod_{j \in J_i} R_{ij} \right) \right) \]

for some \(R_{ij} \in \text{Rel}_n(A)\), some coordinate projection \(\pi\), and some coordinate permutations \(\mu_i\).
Lemma

Suppose that

$$\text{conf}(1, 0, 0) \subseteq \pi \left( \bigcap_{i \in I} \mu_i \left( \prod_{j \in J_i} R_{ij} \right) \right) = S \leq A(\mathcal{M})^m,$$

where $\pi$ is a projection, the $\mu_i$ are permutations, and the $R_{ij}$ are a finite collection of members of $\text{Rel}_n(A(\mathcal{M}))$, and $n < m$. Then $S$ is halting.

Theorem

The following hold for any Minsky machine $\mathcal{M}$.

- If $\mathcal{M}$ does not halt with capacity $m$ then $m < \deg(A(\mathcal{M}))$.
- If $\mathcal{M}$ does not halt then $A(\mathcal{M})$ is not finitely related.

Proof: Suppose that $\deg(A(\mathcal{M})) \leq m$. This implies in particular that $\text{Rel}_m(A(\mathcal{M})) \models S_{m+1}$. By Zadori’s theorem, $S_{m+1}$ can be represented as in the Lemma above, so by that same Lemma it is halting. By the Coding Theorem, this implies that $\mathcal{M}$ halts with capacity $m$, a contradiction.
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Strategy

- The relations $S_m$ witnessed non-entailment when $M$ did not halt. When $M$ does halt, these relations eventually witness the halting.

- Show that for some suitably chosen $k$, we have $\text{Rel}_k(A(M)) \models \text{Rel}_n(A(M))$ for all $n$.

- We proceed by induction on $n$.

- The base case of $n = k$ is trivial.

- We thus endeavor to prove $\text{Rel}_{n-1}(A(M)) \models R$ for $R \in \text{Rel}_n(A(M))$.

- Relations in $\text{Rel}_n(A(M))$ can be divided into 4 different kinds, so we proceed by cases.

- We add operations to handle entailment in each of the different cases: $N_\bullet(w, x, y, z)$, $P(u, v, x, y)$, $H(x, y)$, $N_0(x, y, z)$, $S(x, y, z)$.

- $A(M) = \langle A(M) ; M, M', \land, I, N_\bullet, P, H, N_0, S \rangle$ (final version)
\[ \mathbb{A}(\mathcal{M}) = \langle A(\mathcal{M}) ; M, M', \land, I, N_\bullet, P, H, N_0, S \rangle \]

Case \( R \) is non-computational

- There is an element with 2 \( \bullet \)'s or with non-constant state.
- 2 \( \bullet \)'s: operation \( N_\bullet \) handles entailment.
- Non-constant state: operation \( P \) handles entailment.

Theorem

*If \( m \geq 3 \) and \( R \leq \mathbb{A}(\mathcal{M})^m \) is non-computational then \( \text{Rel}_{m-1}(\mathbb{A}(\mathcal{M})) \models R \).*

Case \( R \) is halting

- \( R \) contains an element of \( \text{conf}(0,0,0) \).
- Any element of \( \text{conf}(0,0,0) \) can be used with operations \( I, H \), and \( N_0 \) to prove entailment.

Theorem

*If \( 3 \leq m \) and \( R \leq \mathbb{A}(\mathcal{M})^m \) is halting then \( \text{Rel}_{m-1}(\mathbb{A}(\mathcal{M})) \models R \).*
We are left to examine computational non-halting $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^n$.

Let’s say that $\mathcal{M}$ halts with capacity $\kappa$.

**Two metrics** (both subsets of $[n]$)

- $\mathcal{D}(\mathbb{R}) =$ “coordinates $i$ such that $\exists r \in R$ with $r(i) = \langle j, \bullet \rangle$”
  
  = “the $\bullet$ (dot) part of $\mathbb{R}$.”

- $\mathcal{N}(\mathbb{R}) =$ “the inherently non-halting part of $\mathbb{R}$” ...
  
  - $\pi_{\mathcal{N}(\mathbb{R})}(\mathbb{R})$ is non-halting,
  
  - if $K = |\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})|$ then $\mathbb{S}_K \leq \mathbb{R}$.

**Case $\mathbb{R}$ is computational and $|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})| > \kappa$**

- $|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})| > \kappa$ then $\mathbb{R}$ contains a halting subalgebra.

- it follows that $\mathbb{R}$ halts!

We thus consider computational non-halting $\mathbb{R}$ with $|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})| \leq \kappa$. ·
Case computational non-halting $\mathbb{R}$ with $|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})| \leq \kappa$

**Theorem**

Assume that $n \geq \kappa + 16$ and

- $\mathbb{R} \leq A(\mathcal{M})^n$ is computational non-halting,
- $|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})| \leq \kappa$,
- $\vdots$ (several technical hypotheses)

Then $\text{Rel}_{n-1}(A(\mathcal{M})) \models \mathbb{R}$.

This completes the case analysis!

**Theorem**

If $\mathcal{M}$ halts with capacity $\kappa$ then $\deg(A(\mathcal{M})) \leq \kappa + 16$.  

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Clones and The Finite Degree Problem

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Conclusion and Open Problems
Theorem

The following are equivalent.

• $M$ halts,
• $\deg(A(M)) < \infty$ (i.e. $A(M)$ is finitely related),
• $M$ halts with capacity at least $\deg(A(M)) - 16$.

Interesting observations

• There are infinitely many $M$ with halting status independent of ZFC.
• There are infinitely many $M$ such that $\deg(A(M)) < \infty$ is independent of ZFC.
• There are finite algebras $A$ that whose finite-relatedness is independent of ZFC.
• $\maxdeg_\sigma(n) = \sup \left\{ \deg(A) \mid A \text{ has signature } \sigma, \deg(A) < \infty, \text{ and } |A| \leq n \right\}$ is not computable.
Finite Generation Problems

Problem
Given relations $\mathcal{R}$, decide if $\mathcal{C} = \text{Pol}(\mathcal{R})$ is finitely generated. That is, decide whether $\mathcal{C} = \text{Clo}(\mathcal{F})$ for some finite set of operations $\mathcal{F}$.

Problem
Given relations $\mathcal{R}$ and operations $\mathcal{F}$, decide whether $\text{Pol}(\mathcal{R}) = \text{Clo}(\mathcal{F})$.
We can modify the definition of $\text{deg}(\cdot)$ to obtain a duality degree: $\text{deg}_\partial(\cdot)$.

**Problem (Finite Duality Degree)**

Decide whether $\text{deg}_\partial(\mathbb{A}) < \infty$ for finite $\mathbb{A}$.

Duality entailment implies usual entailment, so we already have that $\mathbb{A} (\mathcal{M})$ is not finitely duality related when $\mathcal{M}$ does not halt.

**Problem**

If $\mathcal{M}$ halts, is $\text{deg}_\partial(\mathbb{A} (\mathcal{M})) < \infty$?

**Problem**

Given finite $\mathbb{A}$, decide whether $\mathbb{A}$ admits a duality.
The following are equivalent.

- $\mathcal{M}$ halts,
- $\deg(\mathbb{A}(\mathcal{M})) < \infty$ (i.e. $\mathbb{A}(\mathcal{M})$ is finitely related),
- $\mathcal{M}$ halts with capacity at least $\deg(\mathbb{A}(\mathcal{M})) - 16$.

Thank you for your attention.