RESEARCH STATEMENT

HENRI MÜHLE

0. Background

My research belongs to an area of combinatorics, usually entitled Coxeter-Catalan combinatorics, i.e. the combinatorial investigation of families of objects arising from Coxeter groups (or reflection groups in general) such that their cardinality is given by the generalized Catalan numbers defined in (1) below. I am particularly interested in topological and structural properties of these families when equipped with different partial orders. Besides that, I am also interested in combinatorial poset construction, namely in finding new ways to construct partial orders by combining several given posets, and subsequently investigating their combinatorial properties.

0.1. Reflection Groups. A reflection in an Euclidean vector space \( V \) is a unitary transformation of finite order that fixes a hyperplane pointwise. A finite unitary reflection group \( W \) is a finite subgroup of the group of all unitary transformations on \( V \) that is generated by reflections, and these groups have been characterized by G. C. Shephard and J. A. Todd in [48]. If \( W \) acts on an \( n \)-dimensional vector space and can be generated by \( n \) reflections, then we call \( W \) well-generated, and I am mainly working with those well-generated reflection groups.

If we restrict ourselves to the case \( V = \mathbb{R}^n \), then we obtain precisely the finite Coxeter groups classified in [19]. These groups can be defined purely algebraically in terms of a Coxeter system \((W,S)\), where \( S \) is a set of generators of \( W \) having cardinality \( n \) (the so-called rank of \( W \)) and satisfying certain relations. The definition of a Coxeter system does not necessarily restrict to finite groups \( W \) and most of the combinatorial beauty of finite Coxeter groups generalizes to the infinite case.

0.2. Coxeter-Fuß-Catalan Numbers. Every finite reflection group \( W \) possesses a set of algebraic invariants, called the degrees of \( W \). In particular, these are the degrees of certain polynomials generating the ring of \( W \)-invariant polynomials. If we denote these degrees by \( d_1, d_2, \ldots, d_n \) (where their values increase weakly with respect to their indices), then we can define the \( W \)-Catalan numbers by

\[
\text{Cat}(W) = \prod_{i=1}^{n} \frac{d_n + d_i}{d_i},
\]

which was probably first written down by V. Reiner in [45, Remark 2]. In [4], C. Athanasiadis generalized this formula by adding a parameter \( m \), and defined the so-called \( W \)-Fuß-Catalan numbers

\[
\text{Cat}^{(m)}(W) = \prod_{i=1}^{n} \frac{md_n + d_i}{d_i}.
\]

The name of these numbers comes from the fact that in the case where \( W \) is the symmetric group \( \mathfrak{S}_n \), we obtain \( \text{Cat}^{(m)}(\mathfrak{S}_n) = \frac{1}{n} \binom{mn+n}{n-1} \), which are precisely the classical Fuß-Catalan numbers, which first appeared in [53].
Frequently, he special case \( W = \mathfrak{S}_n \), which I will refer to as the type A case, yields a well-known “classical” Catalan object which in fact motivated the study of the general case. Moreover, we refer to families of objects counted by \( \text{Cat}(W) \) as (Coxeter-)Catalan objects, and we refer to families of objects counted by \( \text{Cat}^{(m)}(W) \) as (Coxeter-)Fuß-Catalan objects. By a Fuß-Catalan generalization, I mean a generalization of a (Coxeter-)Catalan object to a (Coxeter-)Fuß-Catalan object.

0.3. Some Partial Orders on Reflection Groups. Let \( T \) denote the set of all reflections of a finite reflection group \( W \). Since \( T \) generates \( W \), every element in \( W \) can be written as a product of reflections, and for \( w \in W \), we denote by \( \ell_T(w) \) the minimal length of such a product yielding \( w \). The first partial order that I am interested in is the absolute order on \( W \), defined by \( u \leq_T v \) if and only if \( \ell_T(v) = \ell_T(u^{-1}v) + \ell_T(u) \) for all \( u, v \in W \).

If \( (W, S) \) is a Coxeter system of finite rank, then \( S \) is another finite generating set of \( W \), and thus—analogously to before—every element in \( W \) can be written as a product of elements on \( S \), and for \( w \in W \), we denote by \( \ell_S(w) \) the minimal length of such a product yielding \( w \). Now, define the (right) weak order on \( W \) by \( u \leq_S v \) if and only if \( \ell_S(v) = \ell_S(u^{-1}v) + \ell_S(u) \), for \( u, v \in W \) such that \( \ell_S \) denotes the minimal length of an element in \( W \) written as a product of elements in \( S \).

Past—Topological Properties of Certain Coxeter-Catalan Posets

In this section I describe my contribution to the research of certain subposets of the previously introduced posets which exhibits nice topological properties of these posets, and allows for an explicit computation of their Möbius function.

The main tool used in this research is lexicographic shellability, introduced by A. Björner in [11]: a poset is lexicographically shellable if it admits a labeling such that in every interval there exists a unique maximal chain, whose label sequence is strictly increasing, and such that the the label sequence of every other maximal chain in this interval is lexicographically larger.

An lexicographically shellable poset enjoys several nice properties having an impact on its topology: the associated order complex is shellable, and homeomorphic to a wedge of spheres (and is thus Cohen-Macaulay), and its Möbius function equals the number of falling chains with respect to the labeling. See [12, 13] for an extensive overview.

1.1. Noncrossing Partitions. From a structural point of view, the poset \((W,\leq_T)\) does not possess too many nice properties in general. However, it contains a very well-behaved subposet, namely if we consider the interval \( \mathsf{NC}(W) = [\varepsilon, \gamma]_T \), where \( \varepsilon \in W \) denotes the identity, and \( \gamma \in W \) is a Coxeter element, then we obtain a lattice, the so-called lattice of \( W \)-noncrossing partitions [8, 10, 15–17]. Interestingly, the lattice \( \mathsf{NC}(W) \) does not depend on the choice of \( \gamma \), see [46]. D. Armstrong has generalized this lattice in [1] by considering \((m+1)\)-tuples \((w_0, w_1, w_2, \ldots, w_m)\) with \( w_i \in \mathsf{NC}(W) \) for \( i \in \{1, 2, \ldots, m\} \) such that \( \gamma = w_0w_1 \cdots w_m \in \mathsf{NC}(W) \) and \( \ell_T(\gamma) = \sum_{i=0}^m \ell_T(w_i) \) under componentwise absolute order on the last \( m \) components. The resulting poset is the poset of m-divisible noncrossing partitions, denoted by \( \mathsf{NC}^{(m)}(W) \). Remarkably, the cardinality of \( \mathsf{NC}^{(m)}(W) \) is given by the \( W \)-Fuß-Catalan numbers defined above, see [6, 9, 10, 18, 20, 45], and the type A case corresponds to m-divisible set partitions of \( \{1, 2, \ldots, mn\} \) under refinement order, considered first in [20].

While it was shown (uniformly) in [1, 5] that \( \mathsf{NC}^{(m)}(W) \) is EL-shellable provided \( W \) is a Coxeter group, I showed the same (in a case-by-case fashion) for the remaining well-generated unitary reflection groups (i.e. the infinite family \( G(d, d, n) \) for \( d, n \geq 3 \), and 20 exceptional groups) [34]. I am currently writing a manuscript on the lexicographic shellability of the poset of multichains of
an arbitrary poset. Another current project (joint with V. Ripoll) is the investigation of the relation between lexicographic shellability of $\mathcal{NC}(W)$ and the Hurwitz action on reduced words of $\gamma$. More precisely, we are investigating to which extent this relationship can be extended to arbitrary finitely presented groups.

1.2. Cambrian Lattices. Another famous Catalan poset is the Tamari lattice $T_n$ introduced in [51]. It was long known that the Tamari lattice can be realized by elements of the symmetric group $\Sigma_n$ [13], but it took some more time until a suitable Coxeter-Catalan generalization of $T_n$ was found by N. Reading: the so-called $\gamma$-Cambrian lattices for $\gamma \in W$ a Coxeter element [40]. Earlier constructions in type $B$ were given by R. Simion and H. Thomas, respectively, [49, 52]. Later, N. Reading realized his $\gamma$-Cambrian lattices in terms of so-called $\gamma$-sortable elements—a Coxeter-Catalan generalization of the 312-avoiding permutations—and it turned out that this can be done for arbitrary (not necessarily finite) Coxeter groups, see [42, 44].

The $\gamma$-Cambrian lattices have since been a popular object of research, since they have a close connection to generalized associahedra, noncrossing partitions, and $W$-clusters [26, 27, 38, 41, 43, 47]. Consequently, many properties of these lattices have been investigated using algebraic or geometric methods, provided that $W$ is a finite Coxeter group: N. Reading showed in [39] that every open interval in a $\gamma$-Cambrian lattice is either spherical or contractible by investigating certain hyperplane arrangements; C. Ingalls and H. Thomas showed in [27] that the $\gamma$-Cambrian lattices are trim and thus lexicographically shellable by using representation theory of quivers; V. Pilaud and C. Stump showed in [38] that the homotopy type of the $\gamma$-Cambrian lattices is that of a simplicial sphere by considering the $\gamma$-Cambrian lattices as flip posets of certain subword complexes.

Together with M. Kallipoliti, I reproved and extended these results in [28] to all Coxeter groups in a uniform way using the realization of the $\gamma$-Cambrian lattices in terms of $\gamma$-sortable elements. Recently, I have extended the result of C. Ingalls and H. Thomas that the $\gamma$-Cambrian lattices associated with a finite Weyl group are trim to all Coxeter groups [36].

1.3. Bruhat Lattices. Another recent tool for determining the topological structure of a lattice, are so-called SB-labelings, introduced by P. Hersh and K. Mészáros in [25]. The existence of an SB-labeling implies that the crosscut complex of the lattice is either a simplex or the boundary of a simplex. Since the crosscut complex is homotopy equivalent to the order complex, we can use this knowledge to determine the Möbius function of the given lattice.

I considered the sortable elements of a Coxeter group with respect to another natural partial order: the Bruhat order, which in principle is a subword order. In this setting we always obtain lattices (even in infinite type). Moreover, these lattices are join-distributive [2], which means that they can be realized as a hierarchy of feasible sets of some antimatroid [21]. In [35], I showed that these lattices admit an SB-labeling, which implies that each interval in these lattices is homotopic to either a sphere or a ball.

Moreover, for some particular Coxeter groups (the so-called coincidental types), there exist Coxeter elements such that the resulting Bruhat lattice is distributive, and coincides with the lattice of order ideals of the corresponding root poset (except for $H_3$) [35]. Inspired by this observation, I investigated in [31] particular Bruhat lattices in type $A$ and $B$, which can be realized as posets of Dyck paths, showed that they form a Heyting algebra, and gave explicit formulas to compute relative pseudocomplements and regular elements in these algebras.
1.4. \textit{m-Tamari Lattices}. A recent Fuß-Catalan generalization of $\mathcal{T}_n$ is the \textit{m-Tamari lattice} $\mathcal{T}_n^{(m)}$ due to F. Bergeron and L.-F. Préville-Ratelle, see [7]. It was shown in [14] that $\mathcal{T}_n^{(m)}$ is an interval in $\mathcal{T}_{mn}$, and thus many of the nice properties of the Tamari lattices generalize to the $m$-Tamari lattices. Nevertheless, I investigated the topology of $\mathcal{T}_n^{(m)}$ in [30], and gave a uniform proof (for all $m$ and $n$) that these lattices are lexicographically shellable, computed their Möbius function, and characterized the spherical and the contractible intervals. Further research in this direction might be the definition of an $m$-Tamari lattice of type $B$, including a suitable Fuß-Catalan generalization of Dyck paths of type $B$.

The $m$-Tamari lattices also play an important role in my current research on poset construction, since they serve as a motivating example for the $m$-cover construction that will be explained in the next section.

\textbf{Present—Poset Construction}

Given a family of posets, there are several constructions known how to construct a single poset from this family, for instance direct or tensor product, cardinal or ordinal sum, Segre or Rees product, and many more. Within the last years, I investigated two new combinatorial construction methods.

\subsection{2.1. Merging Partial Orders.} In joint work with B. Ganter and C. Meschke, I defined for two posets $\mathcal{P}$ and $\mathcal{Q}$ a \textit{merging of $\mathcal{P}$ and $\mathcal{Q}$}, namely a partial order on the disjoint union of $\mathcal{P}$ and $\mathcal{Q}$ such that the restriction to either groundset yields the original partial order again [24]. We completely characterized the possible mergings of $\mathcal{P}$ and $\mathcal{Q}$ in terms of pairs of binary relations between the groundsets, and showed that they possess a natural distributive lattice structure. Subsequently, I gave enumeration formulas for certain special cases [32, 33]. Further research in this direction could involve the investigation how the lattice of mergings of two posets changes, when we modify the underlying posets.

\subsection{2.2. The $m$-Cover Construction.} In recent work with M. Kallipoliti on realizing the $m$-Tamari lattices in terms of $m$-tuples of Dyck paths, we constructed a certain subposet of the $m$-fold direct product of a bounded poset with itself, the so-called \textit{$m$-cover poset}. This poset consists of $m$-tuples of the form $(0^{l_0}, p^{l_1}, q^{l_2})$, where $0$ is the least element of the poset, $p$ and $q$ form a cover relation in this poset, the exponents $l_0, l_1, l_2$ sum up to $m$, and we understand $p^l$ to be the sequence $p, p, \ldots, p$ of length $l$ [29]. We characterized the posets whose $m$-cover poset is a lattice, a left-modular lattice, a trim lattice or a lexicographically shellable lattice. We gave a formula for the cardinality of the $m$-cover poset, and characterized its irreducible elements. When considering the $m$-cover poset of the Tamari lattice, it turns out that this is in general not a poset. However, its so-called \textit{Dedekind-MacNeille completion}, namely the smallest lattice containing this poset, is isomorphic to the $m$-Tamari lattice. An important tool for the proof of this result is a certain decomposition of $m$-Dyck paths into $m$-tuples of Dyck paths, which is conjectured to realize the $m$-Tamari lattice as well, after a certain modification [29]. In the same article, we introduced a family of Fuß-Catalan lattices associated with the dihedral groups which specialize to the corresponding Cambrian lattice in the case $m = 1$. 
Future—Generalizations of the Tamari Lattices

The Tamari lattices, in many different guises, have been central objects of my research so far, along with many generalizations. It is the aim of my future work to pursue further generalizations, which I will describe in the following sections.

3.1. Posets of Generalized Triangulations. The original definition of the type A Cambrian lattices was given in terms of a partial order on the set of triangulations of a convex \((n+2)\)-gon, where the covering relations are given by diagonal flips [40]. A straightforward way to construct a Fuß-Catalan generalization of this approach, is considering \((m+2)\)-angulations of a convex \((mn+2)\)-gon. Then, however, we have several choices of “flipping” diagonals, and first examples suggest that different flipping strategies yield differently behaved posets. A short-time goal is to investigate these generalized flip posets in detail, considering different flipping strategies, and different embeddings of the polygons.

Moreover, it was shown in [23] that triangulations of a polygon can be interpreted as type A clusters, and this connection yields a simplicial complex. In [22], S. Fomin and N. Reading defined a Fuß-Catalan generalization of this complex, the so-called \(m\)-cluster complex for finite Coxeter groups. The corresponding type A case, is connected to the generalized triangulations defined above, and thus a natural research question is to investigate the connection between the \(m\)-cluster complex of the symmetric group and the generalized flip posets from the previous paragraph. Once this case has been understood, a subsequent project will be a generalization of this construction towards type \(B\), namely considering centrally symmetric \((m+2)\)-angulations of a \(2(mn+1)\)-gon, i.e. generalized triangulations fixed under a half turn. The case \(m = 1\) has been investigated in [40, 52].

3.2. “Tamari like” Posets coming from Diagonal Harmonics. The original definition of the \(m\)-Tamari lattices \(T^{(m)}_n\) due to F. Bergeron and L.-F. Préville-Ratelle was motivated by the study of graded Frobenius characteristic of the space of higher diagonal harmonic polynomials. In particular, they proposed a formula in which a sum runs over the intervals in \(T^{(m)}_n\). Once again, the space in question can be seen as a type A case of a more general setting. Generalizations of these spaces exist for all crystallographic Coxeter groups. It is thus a very intriguing task to give a combinatorial description of the posets involved in this more general setting. The resulting posets could then be seen as Coxeter-Catalan generalizations of \(T^{(m)}_n\).

C. Stump, H. Thomas and N. Williams have currently defined a Fuß-Catalan generalization of the Cambrian lattices in terms of generalized subword complexes [50]. However, the \(m\)-Tamari lattices are not obtained as a type A case. Thus the problem of defining \(m\)-Tamari lattices for all Coxeter groups is still open, and my work with M. Kallipoliti [29] is so far the only available approach, which however only provides a partial solution. The approach via the spaces of diagonal harmonics might yield a better understanding of this problem.

Recall that we can define rational Tamari lattices, i.e. posets parametrized by two coprime parameters \(a\) and \(b\), whose cardinality is given by the rational Catalan number

\[
\text{Cat}(a,b) = \frac{(a + b - 1)!}{a! b!},
\]

see for instance [3]. We recover the classical Tamari lattices in the case \(a = n\) and \(b = n + 1\), and the \(m\)-Tamari lattices in the case \(a = n\) and \(b = mn + 1\). While these posets are quite well-behaved, in
the sense that they always form intervals in some Tamari lattice, it might be interesting to see what happens if we drop the condition that \(a\) and \(b\) are coprime.

3.3. Restrictions to Parabolic Quotients. If we consider a Coxeter system \((W, S)\) and pick some \(J \subseteq S\), then we can factorize \(W\) into two parts: a parabolic subgroup \(W_J\) consisting of elements whose reduced words use only letters in \(J\), and a parabolic quotient \(W_J^\perp\) consisting of minimal length representatives of the left cosets of \(W_J\) in \(W\). In his thesis N. Williams proposed an extensive research program to define and investigate Coxeter-Catalan objects, such as noncrossing partitions, nonnesting partitions and sortable elements, for parabolic quotients along with many conjectures [54]. Currently N. Williams and I are investigating the type \(A\) case, which results in the definition of parabolic Tamari lattices [37]. The consequent next steps are the extension of this work to all Coxeter groups, and prove structural and topological properties of these parabolic Cambrian lattices.

References

[16] Thomas Brady and Colum Watt, \(K(\pi, 1)\)'s for Artin Groups of Finite Type, Geometriae Dedicata 94 (2002), 225–250.