Parabolic 231-Avoiding Permutations

Let $S_n$ be the symmetric group on $[n] = \{1, 2, \ldots, n\}$, and let $S = \{a_i\}_{i=1}^{n-1}$ be the set of simple reflections $s_i = (i, i+1)$. The Cayley graph of $S_n$ generated by $S$ may be oriented to form the weak order, which is a lattice. Fix $J = S \setminus \{s_j, s_{j+1}, \ldots, s_k\}$, and let $B(J)$ be the set partition of $[n]$ (whose parts we call $J$-regions)

$$\{1, \ldots, j\}, \{j+1, \ldots, j\}, \ldots, \{j-1, \ldots, j\}, \{j+1, \ldots, n\}.$$

The parabolic quotient $\mathfrak{S}_J$ is the set of $w \in S_n$ whose one-line notation has the form $w = w_1 \cdots < w_{n-1} < w_n$.

Definition 1 A permutation $w \in \mathfrak{S}_J$ is 231-avoiding if there exist no three indices $i < j < k$ of which lie in different $J$-regions, such that $w_j < w_i < w_k$ and $w_k = w_i = 1$. Let $\mathfrak{S}_J^{231}$ denote the set of J-231-avoiding permutations of $\mathfrak{S}_J$.

Theorem 2 For $J \subseteq S$, the restriction of the weak order to $\mathfrak{S}_J^{231}$ forms a lattice. Furthermore, $\mathfrak{T}_J^{231}$ is a lattice quotient of the weak order on $S_n$.

When $J = \emptyset$, we recover the classical Tamari lattice on 231-avoiding permutations.

Parabolic Noncrossing Partitions

Let $P = \{P_1, P_2, \ldots, P_r\}$ be a set partition of $[n]$. A pair $(a, b)$ is a bump of $P$ if $a, b \in P_i$ for some $i \in [s]$ and there is no $c \in P_i$ with $a < c < b$.

Definition 3 A set partition $P$ of $[n]$ is $J$-noncrossing if it satisfies:

(NC1) If $i$ and $j$ lie in the same $J$-region, then they are not contained in the same part of $P$.

(NC2) If two distinct bumps $(i_1, i_2)$ and $(j_1, j_2)$ of $P$ satisfy $i_1 < j_1 < j_2 < i_2$, then either $i_1$ and $j_1$ lie in the same $J$-region or $j_1$ and $i_2$ lie in the same $J$-region.

(NC3) If two distinct bumps $(i_1, i_2)$ and $(j_1, j_2)$ of $P$ satisfy $i_1 < j_1 < j_2 < i_2$, then either $i_1$ and $j_1$ lie in different $J$-regions.

Let $NC_J$ denote the set of all $J$-noncrossing set partitions of $[n]$.

When $J = \emptyset$, we recover the classical noncrossing set partitions.

Example 4 For $J = \{s_1, s_2, s_3, s_4, s_5\}$, \{\{1\}, \{2, 9\}, \{3, 10\}, \{4\}, \{5\}, \{6, 8\}, \{7\}\} \in NC_J.

Parabolic Nonnesting Partitions

Definition 5 A set partition $P$ of $[n]$ is $J$-nonnesting if it satisfies:

(NN1) If $i$ and $j$ lie in the same $J$-region, then they are not in the same part of $P$.

(NN2) If $(i_1, i_2)$ and $(j_1, j_2)$ are two distinct bumps of $P$, then it is not the case that $i_1 < j_2 < i_2$.

Let $NN_J$ denote the set of all $J$-nonnesting partitions of $[n]$.

When $J = \emptyset$, we recover the classical nonnesting set partitions.

Example 6 For $J = \{s_1, s_2, s_3, s_4, s_5\}$, \{\{1\}, \{2\}, \{3\}, \{4, 8\}, \{6, 10\}, \{7\}, \{9\}\} \in NN_J.

Parabolic Catalan Objects are Equinumerous

Although we no longer have a product formula for $|\mathfrak{S}_J^{(231)}|$, our parabolic generalizations remain in bijection, generalizing the situation when $J = \emptyset$.

Theorem 7 For $n > 0$ and $J \subseteq S$, we have $|\mathfrak{S}_J^{(231)}| = |NC_J| = |NN_J|$.

From $\mathfrak{S}_J^{(231)}$ to $NC_J$

A permutation $w \in \mathfrak{S}_J^{(231)}$ corresponds to the $J$-noncrossing partition $P \in NC_J$ whose bumps are determined by the descents of $w$.

Example 8 For $n = 10$ and $J = \{s_1, s_2, s_3, s_4, s_5\}$, $1 \hspace{1mm} 7 \hspace{1mm} 9 \hspace{1mm} 10 \hspace{1mm} | \hspace{1mm} 2 \hspace{1mm} 5 \hspace{1mm} | \hspace{1mm} 3 \hspace{1mm} 1 \hspace{1mm} | \hspace{1mm} 6 \hspace{1mm} 8 \hspace{1mm} \in \mathfrak{S}_J^{(231)}$ gives $\in NC_J.$

From $NC_J$ to $NN_J$

A $J$-noncrossing partition $P$ corresponds to the $J$-nonnesting partition $P'$, in which the minimal elements outside $P'$ are determined by the bumps of $P$.

Example 9 For $n = 10$ and $J = \{s_1, s_2, s_3, s_4, s_5\}$, we compute

Example: $\mathfrak{S}_J, J = \{s_2\}$

Outlook

We can generalize the definition of $J$-231-avoidable elements of $\mathfrak{S}_J$ to parabolic quotients of any finite Coxeter group and to any Coxeter element. The definition of parabolic noncrossing and nonnesting partitions—as well as the $\alpha$-cluster complex—can also be generalized, but—in contrast to the classical case when $J = \emptyset$—the four sets are not always equinumerous.