**Day’s Doubling Construction**

Let $P = (P, \leq)$ be a poset and let $2$ be the chain of length $2$ whose elements are $0$ and $1$. For $I \subseteq P$, define $P_I = \{ x \mid x \leq y \text{ for some } y \in I \}$. The **doubling** of $P$ by $I$ is the subposet $P[I]$ of $P \times 2$ given by the ground set $\{ P_I \times \{ 0 \} \} \cup \{ ((P \setminus P_I) \cup I) \times \{ 1 \} \}$.

**The Alternate Order**

Let $P = (P, \leq)$ be a congruence-uniform lattice. For $x \in P$, define $x_i = x \wedge y$, and $\Psi(x) = \{ \lambda(x, y) \mid x_i \leq u \wedge v \leq x \}$. The **alternate order** of $P$ is the poset $\text{Alt}(P) = (P, \leq)$ determined by the order relation $x \subseteq y$ if and only if $\Psi(x) \subseteq \Psi(y)$.

**Problem 1** (N. Reading, 2016)

For which congruence-uniform lattices is their alternate order again a lattice?

**The Motivation**

Let $A$ be a simplicial hyperplane arrangement, and fix a base region $B$. The **poset of regions** $\text{Alt}(A, B)$ is the reflexive and transitive closure of the adjacency graph of the regions of $A$ oriented away from $B$.

**Example**

![Example Diagram]

**A Necessary Condition**

**Theorem 5** (N. Reading, 2017)

Let $P$ be a congruence-uniform lattice. If $\text{Alt}(P)$ is a lattice, then $P$ is spherical.

**Sketch of proof**: use meet-semidistributivity of $P$ and Theorem 3 to show that $\text{Alt}(P)$ has a greatest element if and only if $P$ is spherical.

**Another Example**

![Another Example Diagram]

**A Particular Doubling**

Let $P = (P, \leq)$ be a lattice. An element $j \in P \setminus \{ \hat{0}, \hat{1} \}$ is **join-irreducible** if $j = x \vee y$ implies $j \in \{ x, y \}$.

**Proposition 6**

Every meet-semidistributive lattice $P$ satisfies $\mu_P(\hat{0}, \hat{1}) = -1$. A meet-semidistributive lattice $P$ is **spherical** if $\mu_P(\hat{0}, \hat{1}) \neq 0$.

**A Special Hyperplane Arrangement**

![A Special Hyperplane Arrangement]

**The Crosscut Theorem**

Let $P = (P, \leq)$ be a lattice with least element $\hat{0}$ and greatest element $\hat{1}$. An antichain $C \subseteq P \setminus \{ \hat{0}, \hat{1} \}$ is a **crosscut** if every maximal chain of $P$ intersects $C$.

A crosscut $C$ is **spanning** if $\forall C = \hat{1}$ and $\hat{1} C = \hat{0}$.

**Theorem 3** (G.-C. Rota, 1964)

Let $P = (P, \leq)$ be a lattice, and let $C \subseteq P$ be a crosscut. We have $\mu_P(\hat{0}, \hat{1}) = \sum_{X \subseteq C \text{ spanning}} (-1)^{|X|}$.

**Shards of Hyperplanes**

Let $X$ be an intersection of hyperplanes of $A$ of codimension $2$. The regions containing $X$ form a polyhedral complex $P(A, B)$ with a greatest element $\hat{1}$.

The **bounder hyperplanes** of $Q$ that contain $X$ “cut” all the other hyperplanes containing $X$. All these cuts split the hyperplanes of $A$ into **shards**.

**Example**

![Shards of Hyperplanes Example Diagram]

**The Möbius Function**

The **Möbius function** of a poset $P$ is the function $\mu_P$ defined recursively by:

$$\mu_P(x, y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{z < x \leq y} \mu_P(x, z), & \text{if } x < y, \\ 0, & \text{otherwise}. \end{cases}$$

**Spherical Meet-Semidistributive Lattices**

**Proposition 4**

Every meet-semidistributive lattice $P$ satisfies $\mu_P(\hat{0}, \hat{1}) = -1$. A meet-semidistributive lattice $P$ is spherical if $\mu_P(\hat{0}, \hat{1}) \neq 0$.

**The Intersection Property**

A congruence-uniform lattice $P = (P, \leq)$ has the intersection property if for every $x, y \in P$ there exists some $z \in P$ such that $\Psi(x) \cap \Psi(y) = \Psi(z)$.

**Proposition 8**

A congruence-uniform lattice $P = (P, \leq)$ has the intersection property if for every $x, y \in P$ there exists some $z \in P$ such that $\Psi(x) \cap \Psi(y) = \Psi(z)$.

**Proposition 9**

Which congruence-uniform lattices have the intersection property?

**Problem 9**

Which congruence-uniform lattices have the intersection property?

**Problem 10**

Find a spherical congruence-uniform lattice $P$ without the intersection property for which $\text{Alt}(P)$ is a lattice.