On the Topology of the Cambrian Semilattices
Myrto Kallipoliti and Henri Mühle
Fakultät für Mathematik, Universität Wien, 1090 Vienna, Austria

Cambrian Semilattices

For an arbitrary Coxeter group \( W \) and an arbitrary Coxeter element \( \gamma \in W \), Reading and Speyer defined the \( \gamma \)-Cambrian semilattice \( C_\gamma \) as a sublattice of the weak order semilattice. Cambrian semilattices constitute generalizations of the Tamari lattice \( T_n \), to which they reduce when \( W \) is the symmetric group \( S_n \) and \( \gamma \) is the long cycle \( \gamma = (1 \, 2 \cdots n) \).

What is known - The Finite Case

If \( W \) is a finite Coxeter group, and \( \gamma \in W \) is a Coxeter element, then \( C_\gamma \) is a lattice. Considering its topological properties it is known that:
- \( C_\gamma \) is Cohen-Macaulay (in fact it is EL-shellable), and
- every open interval of \( C_\gamma \) is either contractible or spherical.

Definitions

Let \( W \) be a Coxeter group of rank \( n \) with simple generators \( s_1, s_2, \ldots, s_n \) and let \( \gamma = s_1 s_2 \cdots s_k \in W \) be a Coxeter element of \( W \). Let \( \gamma^\infty = s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots \).

Every \( w \in W \) can be written as a subword of \( \gamma^\infty \), in the form
\[
w = s_1^{\delta_{11}} s_2^{\delta_{12}} \cdots s_n^{\delta_{1n}} s_1^{\delta_{21}} s_2^{\delta_{22}} \cdots s_n^{\delta_{2n}} \cdots,
\]
where \( \delta_{ij} \in \{0, 1\} \) and \( k \geq 0 \).

- \( i \)-th block of \( w \): the set \( b_i(w) = \{ s_j \mid \delta_{ij} = 1 \} \)
- \( \gamma \)-sorting word of \( w \): the lexicographically first word of \( \gamma^\infty \) among all reduced words for \( w \)
- \( \gamma \)-sortable element: some \( w \in W \) such that the \( \gamma \)-sorting word of \( w \) satisfies \( b_i(w) \supseteq b_j(w) \supseteq \cdots \supseteq b_k(w) \)
- \( \gamma \)-Cambrian semilattice \( C_\gamma \): the semi-lattice of the weak-order semilattice consisting of all \( \gamma \)-sortable elements

Example - \( \gamma \)-Sorting Words

Let \( W = S_4 \), generated by \( s_i = (i \, i+1) \) for \( i \in \{1, 2, 3\} \), and let \( \gamma = s_1 s_2 s_3 \). The following are reduced words of the same element \( w \in W \):
\[
w_1 = s_1 s_2 s_3 s_1 s_2 s_3, \quad w_2 = s_1 s_2 s_3 s_1 s_2, \quad w_3 = s_1 s_2 s_3 s_2, \quad w_4 = s_1 s_2 s_3 s_2 s_3, \quad w_5 = s_2 s_3 s_1 s_2.
\]
The \( \gamma \)-sorting word of \( w_1 \) is \( w_1' \), and we have \( b_1(w_1) = \{ s_1, s_2, s_3 \} \) and \( b_2(w_1) = \{ s_1, s_2 \} \), with \( b_1(w_1) \supseteq b_2(w_1) \).

Example - \( \gamma \)-Cambrian Semilattices

A classical result on EL-shellable posets states that the dimension of the \( k \)-th homology group of the corresponding truncated order complex is given by the number of falling maximal chains of length \( k + 2 \) (with respect to the EL-labeling).

Using induction on rank and length and the key lemma, we show that there exists at most one falling maximal chain in every closed interval of \( C_\gamma \).