Symmetric Chain Decompositions and the Strong Sperner Property for Noncrossing Partition Lattices

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Sperner’s Theorem

\[ [n] = \{1, 2, \ldots, n\} \text{ for } n \in \mathbb{N} \]

- **antichain**: set of pairwise incomparable subsets of \([n]\)

**Theorem (E. Sperner, 1928)**

*The maximal size of an antichain of \([n]\) is \(\binom{n}{\lfloor \frac{n}{2} \rfloor}\).*
**Sperner’s Theorem**

- **k-family**: family of subsets of \([n]\) that can be written as a union of at most \(k\) antichains

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**Theorem (P. Erdős, 1945)**

The maximal size of a \(k\)-family of \([n]\) is the sum of the \(k\) largest binomial coefficients.
A Generalization

- **poset perspective:**
  - antichain of \([n]\) \(\longleftrightarrow\) antichain in the Boolean lattice \(B_n\)
  - binomial coefficients \(\longleftrightarrow\) rank numbers of \(B_n\)

- \(\mathcal{P}\) .. graded poset of rank \(n\)
- **\(k\)-Sperner:** size of a \(k\)-family does not exceed sum of \(k\) largest rank numbers
- **strongly Sperner:** \(k\)-Sperner for all \(k \leq n\)
Motivation

Symmetric Chain Decompositions

NCP
Complex Reflection Groups
Noncrossing Partitions

SCD of $\mathcal{NC}_G(d,d,n)$
The Group $G(d,d,n)$
A First Decomposition
A Second Decomposition

SSP of $\mathcal{NC}_W$

A Generalization

- **poset perspective:**
  - antichain of $[n] \leftrightarrow$ antichain in the Boolean lattice $\mathcal{B}_n$
  - binomial coefficients $\leftrightarrow$ rank numbers of $\mathcal{B}_n$

- $\mathcal{P}$ .. graded poset of rank $n$
- **$k$-Sperner:** size of a $k$-family does not exceed sum of $k$ largest rank numbers
- **strongly Sperner:** $k$-Sperner for all $k \leq n$
Examples

- a strongly Sperner poset
Examples

- a Sperner poset that is not 2-Sperner
Examples

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Examples

- a 2-Sperner poset that is not Sperner
Examples

- a 2-Sperner poset that is not Sperner
Examples

- **strongly Sperner posets:**
  - Boolean lattices
  - divisor lattices
  - lattices of noncrossing set partitions
  - Bruhat posets of finite Coxeter groups
  - weak order lattice of $H_3$

- **non-Sperner posets:**
  - lattices of set partitions
  - geometric lattices
Examples

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- **non-Sperner posets:**
  - lattices of set partitions (of very large sets...)
  - geometric lattices (certain bond lattices of graphs)
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   - Complex Reflection Groups
   - Noncrossing Partitions
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   - The Group $G(d,d,n)$
   - A First Decomposition
   - A Second Decomposition
5. Strong Sperner Property of $\mathcal{NC}_W$
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Poset Decompositions

- \( \mathcal{P} \) .. graded poset of rank \( n \)
- **decomposition**: partition of \( \mathcal{P} \) into connected subposets
**Motivation**

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NC>P

**SCD of**

\[ \mathcal{N} G(d,d,n) \]

The Group \( G(d,d,n) \)

A First Decomposition

A Second Decomposition

**SSP of**

\[ \mathcal{N} \mathcal{W} \]

\( \mathcal{P} \) .. graded poset of rank \( n \)

**symmetric decomposition**: parts sit in \( \mathcal{P} \) symmetrically, i.e. match minimal and maximal elements so that ranks add up to \( n \)
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Poset Decompositions

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Poset Decompositions

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![Diagram showing the poset decomposition](image)
- $\mathcal{P}$ .. graded poset of rank $n$
- **symmetric chain decomposition**: symmetric decomposition where parts are chains
Symmetric Chain Decompositions

- $\mathcal{P}$ .. graded poset of rank $n$

**Theorem**

If $\mathcal{P}$ admits a symmetric chain decomposition, then $\mathcal{P}$ is strongly Sperner.
Symmetric Chain Decompositions

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SSP of $\mathcal{N}C_W$

- $\mathcal{P}$ .. graded poset of rank $n$

Theorem

If $\mathcal{P}$ and $\mathcal{Q}$ admit a symmetric chain decomposition, then so does $\mathcal{P} \times \mathcal{Q}$. 
Symmetric Chain Decompositions

- $\mathcal{P}$ .. graded poset of rank $n$; $N_i$ .. size of $i^{th}$ rank
- **rank-symmetric**: $N_i = N_{n-i}$
- **rank-unimodal**: $N_0 \leq \cdots \leq N_j \geq \cdots \geq N_n$
- **Peck**: strongly Sperner, rank-symmetric, rank-unimodal

**Theorem**

If $\mathcal{P}$ admits a symmetric chain decomposition, then $\mathcal{P}$ is Peck.
Symmetric Chain Decompositions

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**Theorem**

If \( \mathcal{P} \) and \( \mathcal{Q} \) are Peck, then so is \( \mathcal{P} \times \mathcal{Q} \).
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Complex Reflection Groups

- $V$ .. $n$-dimensional unitary vector space
- **complex reflection**: unitary transformation of finite order that fixes a hyperplane
- **reflecting hyperplane**: fixed space of a reflection
- **complex reflection group**: finite subgroup of $U(V)$ generated by reflections
- **irreducible**: does not preserve a proper subspace of $V$
- **rank**: codimension of fixed space
- **well-generated**: irreducible, rank equals minimal number of generators
- **parabolic subgroup**: maximal subgroup that fixes a proper subspace of $V$
Classification of Irreducible Complex Reflection Groups

- one infinite family $G(de,e,n)$:
  - monomial $(n \times n)$-matrices
  - non-zero entries are $(de)^{th}$ roots of unity
  - product of non-zero entries is $d^{th}$ root of unity
- 34 exceptional groups $G_4, G_5, \ldots, G_{37}$
Classification of Irreducible Complex Reflection Groups

- one infinite family $G(\text{de},e,n)$:
  - monomial $n \times n$-matrices
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  - product of non-zero entries is $d^\text{th}$ root of unity

- 34 exceptional groups $G_4, G_5, \ldots, G_{37}$

- well-generated complex reflection groups:
  - $G(1,1,n), n \geq 1$
  - $G(d,1,n), d \geq 2, n \geq 1$
  - $G(d,d,n), d,n \geq 2$
  - 26 exceptional groups
Classification of Irreducible Complex Reflection Groups

- one infinite family $G(\deg, e, n)$:
  - monomial $(n \times n)$-matrices
  - non-zero entries are $(\deg)^{th}$ roots of unity
  - product of non-zero entries is $d^{th}$ root of unity

- 34 exceptional groups $G_4, G_5, \ldots, G_{37}$

- finite Coxeter groups:
  - $G(1, 1, n) \cong A_{n-1}$
  - $G(2, 1, n) \cong B_n$
  - $G(2, 2, n) \cong D_n$
  - $G(d, d, 2) \cong I_2(d)$
  - $G_{24} = H_3, G_{28} = F_4, G_{30} = H_4, G_{35} = E_6, G_{36} = E_7, G_{37} = E_8$
A Distinctive Property

- **degrees**: degrees of a homogeneous choice of generators of the invariant algebra
- usually denoted by $d_1 \leq d_2 \leq \cdots \leq d_n$
- **Coxeter number**: largest degree $h = d_n$

Theorem (G. C. Shephard & J. A. Todd, 1954; C. Chevalley, 1955)

A finite group $G$ is a complex reflection group if and only if its algebra of invariant complex polynomials is a polynomial algebra.
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Regular Elements

- **regular vector**: vector that does not lie in a reflecting hyperplane
- **ζ-regular element**: element with eigenvalue ζ so that the corresponding eigenspace contains a regular vector
- **regular number**: multiplicative order of ζ
- **Coxeter element**: ζ-regular element of order h, where ζ is a $h^{th}$ root of unity
Regular Elements

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**Theorem (G. Lehrer & T. A. Springer, 1999)**

If \( W \) is a well-generated complex reflection group, then \( h \) is a regular number.
regular vector: vector that does not lie in a reflecting hyperplane

ζ-regular element: element with eigenvalue ζ so that the corresponding eigenspace contains a regular vector

regular number: multiplicative order of ζ

Coxeter element: ζ-regular element of order $h$, where ζ is a $h^{th}$ root of unity

Theorem (G. Lehrer & T. A. Springer, 1999)

If $W$ is a well-generated complex reflection group, then $h$ is a regular number.
Noncrossing Partitions

- \( W \) .. complex reflection group; \( T \) .. reflections of \( W \); \( c \) .. Coxeter element
- **absolute length:** \( \ell_T(w) = \min \{k \mid w = t_1 t_2 \cdots t_k, t_i \in T\} \)
- **absolute order:** \( u \leq_T v \) if and only if
  \[
  \ell_T(v) = \ell_T(u) + \ell_T(u^{-1} v)
  \]
- \( W \)-noncrossing partitions:
  \[
  NC_W(c) = \{w \in W \mid w \leq_T c\}
  \]
- write \( NC_W(c) = (NC_W(c), \leq_T) \)
Noncrossing Partitions

Theorem (V. Reiner, V. Ripoll & C. Stump, 2015)

For any well-generated complex reflection group $W$, and any two Coxeter elements $c, c' \in W$ we have $\mathcal{NC}_W(c) \cong \mathcal{NC}_W(c')$.


The poset $\mathcal{NC}_W$ is a lattice for any well-generated complex reflection group $W$. 

SCD and SSP for NCP
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Catalan Numbers

- **W-Catalan number:**

\[ \text{Cat}_W = \prod_{i=1}^{n} \frac{d_i + h}{d_i} \]


We have \(|\text{NC}_W| = \text{Cat}_W\) for any well-generated complex reflection group \(W\).
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Symmetric Chain Decompositions of $\mathcal{NC}_G(1,1,n)$

- $W = G(1,1,n) \cong S_n; \ T \ldots \text{transpositions}; \ c = (1 \ 2 \ \ldots \ n)$
- $\mathcal{NC}_G(1,1,n)(c)$ is isomorphic to the lattice of noncrossing set partitions of $[n]$
- $R_k = \{w \in \mathcal{NC}_G(1,1,n)(c) \mid w(1) = k\}, \ R_k = (R_k, \leq_T)$
Symmetric Chain Decompositions of $\mathcal{NC}_G(1,1,n)$

- $W = G(1,1,n) \cong \mathfrak{S}_n$; $T$ ... transpositions; $c = (1 2 \ldots n)$
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- $R_k = \{ w \in \mathcal{NC}_G(1,1,n)(c) \mid w(1) = k \}$, $R_k = (R_k, \leq_T)$
- $\cup$ ... disjoint set union; $2$ ... $2$-chain

**Lemma (R. Simion & D. Ullmann, 1991)**

We have $R_1 \uplus R_2 \cong 2 \times \mathcal{NC}_G(1,1,n-1)$, and $R_i \cong \mathcal{NC}_G(1,1,i-2) \times \mathcal{NC}_G(1,1,n-i+1)$ whenever $3 \leq i \leq n$. Moreover, this decomposition is symmetric.
Symmetric Chain Decompositions of $\mathcal{NC}_G(1,1,n)$

- $W = G(1,1,n) \cong \mathfrak{S}_n$; $T$. .. transpositions; $c = (1 2 \ldots n)$
- $\mathcal{NC}_G(1,1,n)(c)$ is isomorphic to the lattice of noncrossing set partitions of $[n]$
- $R_k = \{w \in \mathcal{NC}_G(1,1,n)(c) \mid w(1) = k\}$, $R_k = (R_k, \leq_T)$
- $\uplus$ .. disjoint set union; $2$ .. 2-chain

Theorem (R. Simion & D. Ullmann, 1991)

The lattice $\mathcal{NC}_G(1,1,n)$ admits a symmetric chain decomposition for each $n \geq 1$. 
Example: $\mathcal{NC}_{S_4}((1\ 2\ 3\ 4))$
Example: $\mathcal{NC}_{\mathfrak{S}_4}(1234)$
Example: $\mathcal{NC}_{\mathfrak{S}_4}((1 2 3 4))$
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The Groups $G(d,d,n)$, $d,n \geq 2$

- subgroups of $\mathfrak{S}_{dn}$, permuting elements of
  \[ \{ 1^{(0)}, \ldots, n^{(0)}, 1^{(1)}, \ldots, n^{(1)}, \ldots, 1^{(d-1)}, \ldots, n^{(d-1)} \} \]

- $w \in G(d,d,n)$ satisfies $w(k^{(s)}) = \pi(k)^{(s+t_k)}$
  \[ \sum_{k=1}^{n} t_k \equiv 0 \pmod{d} \]
  \[ \pi \in \mathfrak{S}_n, \text{ and } t_k \text{ depends on } w \text{ and } k \]
The Groups $G(d,d,n), \, d, n \geq 2$

- elements can be decomposed into “cycles”:

$$\left(\left(\begin{array}{c} k_1^{(t_1)} & \ldots & k_r^{(t_r)} \end{array}\right) \right) = \left(\begin{array}{c} k_1^{(t_1)} & \ldots & k_r^{(t_r)} \end{array}\right) \left(\begin{array}{c} k_1^{(t_1+1)} & \ldots & k_r^{(t_r+1)} \end{array}\right) \cdots \left(\begin{array}{c} k_1^{(t_1+d-1)} & \ldots & k_r^{(t_r+d-1)} \end{array}\right),$$

and

$$\left[ k_1^{(t_1)} \ldots k_r^{(t_r)} \right]_s = \left(\begin{array}{c} k_1^{(t_1)} & \ldots & k_r^{(t_r)} & k_1^{(t_1+s)} & \ldots & k_1^{(t_1+(d-1)s)} \end{array}\right) \cdots \left(\begin{array}{c} k_r^{(t_r+s)} & \ldots & k_1^{(t_1(d-1)s)} & \ldots & k_r^{(t_r+(d-1)s)} \end{array}\right).$$
The Lattices $\mathcal{NC}_{G(d,d,n)}, d, n \geq 2$

- **Coxeter element** $c = \left[ 1^{(0)} \ldots (n-1)^{(0)} \right]_1 \left[ n^{(0)} \right]_{d-1}$

- **Matrix representation:**

$$c = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & \zeta_d & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \zeta_d^{d-1}
\end{pmatrix},$$

where $\zeta = e^{2\pi \sqrt{-1}/d}$
The Lattices $\mathcal{NC}_{G(d,d,n)}$, $d, n \geq 2$

Proposition (‡, 2015)

For $d, n \geq 2$, the atoms in $\mathcal{NC}_{G(d,d,n)}(c)$ are of one of the following forms:

- $\left(\left(a^{(0)} b^{(s)}\right)\right)$ for $1 \leq a < b < n$ and $s \in \{0, d - 1\}$, or
- $\left(\left(a^{(0)} n^{(s)}\right)\right)$ for $1 \leq a < n$ and $0 \leq s < d$. 
The Lattices $\mathcal{NC}_{G(d,d,n)}$, $d, n \geq 2$

**Proposition (.handleChange 2015)**

For $d, n \geq 2$, the coatoms in $\mathcal{NC}_{G(d,d,n)}(c)$ are of one of the following forms:

- $\left[ 1^{(0)} \ldots a^{(0)} (b + 1)^{(0)} \ldots (n - 1)^{(0)} \right]_1 \left[ n^{(0)} \right]_{d-1}$
- $\left((a + 1)^{(0)} \ldots b^{(0)}\right)$ for $1 \leq a < b < n$,
- $\left((1^{(0)} \ldots a^{(0)} (b + 1)^{(d-1)} \ldots (n - 1)^{(d-1)})\right)$
- $\left((1^{(0)} \ldots a^{(0)} (b + 1)^{(d-1)} \ldots (n - 1)^{(d-1)})\right)$
- for $1 \leq a < b < n$, or
- $\left((1^{(0)} \ldots a^{(0)} n^{(s-1)} (a + 1)^{(d-1)} \ldots (n - 1)^{(d-1)})\right)$ for $1 \leq a < n$ and $0 \leq s < d$. 


Example: $d = 5, n = 3$
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A First Decomposition

\[ R_k^{(s)} = \left\{ w \in \mathcal{NC}_{G(d,d,n)}(c) \mid w(1^{(0)}) = k^{(s)} \right\} \]

\[ \mathcal{R}_k^{(s)} = \left( R_k^{(s)}, \leq_T \right) \]
A First Decomposition

\[ R_k^{(s)} = \{ w \in NC_{G(d,d,n)}(c) | w(1^{(0)}) = k^{(s)} \} \]

\[ R_k^{(s)} = (R_k^{(s)}, \leq_T) \]

**Lemma (†, 2015)**

The sets \( R_1^{(s)} \) and \( R_k^{(s')} \) are empty for \( 2 \leq s < d \) as well as \( 2 \leq k < n \) and \( 1 \leq s' < d - 1 \).
A First Decomposition

$$R_k^{(s)} = \left\{ w \in \mathcal{NC}_{G(d,d,n)}(c) \mid w\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\mathcal{R}_k^{(s)} = \left( R_k^{(s)}, \leq_T \right)$$

**Lemma (endir, 2015)**

The poset $\mathcal{R}_1^{(0)} \sqcup \mathcal{R}_2^{(0)}$ is isomorphic to $2 \times \mathcal{NC}_{G(d,d,n-1)}$. Moreover, its least element has length 0, and its greatest element has length $n$. 
A First Decomposition

\[ R_k^{(s)} = \left\{ w \in \mathcal{NC}_{G(d,d,n)}(c) \mid w(1^{(0)}) = k^{(s)} \right\} \]

\[ \mathcal{R}_k^{(s)} = \left( R_k^{(s)}, \leq_T \right) \]

Lemma (✱, 2015)

The poset \( \mathcal{R}_n^{(s)} \) is isomorphic to \( \mathcal{NC}_{G(1,1,n-1)} \) for \( 0 \leq s < d \). Moreover, its least element has length 1, and its greatest element has length \( n - 1 \).
A First Decomposition

\[ R_k^{(s)} = \left\{ w \in \mathcal{NC}_{G(d,d,n)}(c) \mid w\left(1^{(0)}\right) = k^{(s)} \right\} \]

\[ R_k^{(s)} = \left( R_k^{(s)}, \leq_T \right) \]

**Lemma (∉, 2015)**

The poset \( R_i^{(0)} \) is isomorphic to \( \mathcal{NC}_{G(d,d,n-i+1)} \times \mathcal{NC}_{G(1,1,i-2)} \) whenever \( 3 \leq i < n \). Moreover, its least element has length 1, and its greatest element has length \( n - 1 \).
The poset $\mathcal{R}^{(d-1)}_i$ is isomorphic to $\mathcal{NC}_{G(1,1,n-i)} \times \mathcal{NC}_{G(d,d,i-1)}$ whenever $3 \leq i < n$. Moreover, its least element has length $1$, and its greatest element has length $n - 1$. 

Lemma (��, 2015)
A First Decomposition

\[ R_k^{(s)} = \left\{ w \in \mathcal{N}C_{G(d,d,n)}(c) \mid w(1^{(0)}) = k^{(s)} \right\} \]

\[ R_k^{(s)} = \left( R_k^{(s)}, \leq_T \right) \]

Lemma (®, 2015)

The poset \( R_1^{(1)} \) is isomorphic to \( \mathcal{N}C_{G(1,1,n-2)} \). Moreover, its least element has length 2, and its greatest element has length \( n - 1 \).
A First Decomposition

\[ R_k^{(s)} = \left\{ w \in \mathcal{NC}_{G(d,d,n)}(c) \mid w(1^{(0)}) = k^{(s)} \right\} \]

\[ \mathcal{R}_k^{(s)} = \left( R_k^{(s)}, \leq_T \right) \]

Lemma ())? 2015

The poset \( \mathcal{R}_2^{(d-1)} \) is isomorphic to \( \mathcal{NC}_{G(1,1,n-2)} \). Moreover, its least element has length 1, and its greatest element has length \( n - 2 \).
Example: $d = 5, n = 3$
Example: $d = 5, n = 3$
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   - Complex Reflection Groups
   - Noncrossing Partitions
4. Symmetric Chain Decompositions of $\mathcal{NC}_{G(d,d,n)}$
   - The Group $G(d,d,n)$
   - A First Decomposition
   - A Second Decomposition
5. Strong Sperner Property of $\mathcal{NC}_W$
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

\[ f_1 : R_1^{(1)} \to \mathcal{NC}_{G(d,d,n)}(c), \quad x \mapsto \left( \left( \begin{array}{c} 1^0 \\ n^{(d-2)} \end{array} \right) \right) x \]
A Second Decomposition

- bad parts: \( R_1^{(1)} \) and \( R_2^{(d-1)} \)
- consider the map
  \[
  f_1 : R_1^{(1)} \rightarrow R_n^{(d-1)}, \quad x \mapsto \left( \left( \begin{array}{c} 1 \end{array} \right) n^{(d-2)} \right) x
  \]
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  \[ f_1 : R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto (1(0) \ n^{(d-2)})x \]
- this map is an injective involution
- its image consists of permutations $w \in R_n^{(d-1)}$ with
  \[ w(n^{(d-1)}) = 1^{(0)} \]
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  \[ f_1 : R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(1^{(0)} \cdot n^{(d-2)}\right)x \]
- this map is an injective involution
- its image is the interval
  \[ \left[\left(1^{(0)} \cdot n^{(d-1)}\right), \left(1^{(0)} \cdot n^{(d-1)}\right) \left(2^{(0)} \ldots (n-1)^{(0)}\right)\right]_T \]
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  \[ f_1 : R_1^{(1)} \rightarrow R_n^{(d-1)}, \quad x \mapsto \left(\binom{1}{0} n^{(d-2)}\right)x \]

**Lemma (M, 2015)**

The interval \( \left(f_1 \left(R_1^{(1)}\right), \leq_T\right) \) is isomorphic to \( \mathcal{NC}_{G(1,1,n-2)} \).
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  \[ f_1 : R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^0\right) n^{(d-2)}\right)x \]
- define $D_1 = R_1^{(1)} \cup f_1 \left(R_1^{(1)}\right)$, and $D_1 = (D_1, \leq_T)$
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  \[ f_1 : R_1^{(1)} \rightarrow R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)}, n^{(d-2)}\right)\right)x \]
- define $D_1 = R_1^{(1)} \cup f_1\left(R_1^{(1)}\right)$, and $D_1 = (D_1, \leq_T)$

Lemma (\cite{Muehle}, 2015)

The poset $D_1$ is isomorphic to $2 \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n - 1$. 
bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

consider the map

$$f_2 : R_2^{(d-1)} \rightarrow NC_{G(d,d,n)}(c), \quad x \mapsto \left(\left[\begin{array}{c}2^{(0)} \ n^{(0)}\end{array}\right]\right)x$$
bad parts: \( R_1^{(1)} \) and \( R_2^{(d-1)} \)

consider the map

\[
f_2 : R_2^{(d-1)} \rightarrow R_n^{(d-1)}, \quad x \mapsto \left( \begin{pmatrix} 2^{(0)} & n^{(0)} \end{pmatrix} \right) x
\]
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  $$f_2 : R_2^{(d-1)} \rightarrow R_n^{(d-1)}, \quad x \mapsto \left( \left( 2^{(0)} n^{(0)} \right) \right) x$$
- this map is an injective involution
- its image consists of permutations $w \in R_n^{(d-1)}$ with
  $$w \left( n^{(d-1)} \right) = 2^{(d-1)}$$
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  \[ f_2 : R_2^{(d-1)} \rightarrow R_n^{(d-1)}, \quad x \mapsto \left(\begin{array}{c} 2^{(0)} \\ n^{(0)} \end{array}\right)x \]
- this map is an injective involution
- its image is the interval
  \[
  \left[ \left(\begin{array}{c} 1^{(0)} \\ n^{(d-1)} \\ 2^{(d-1)} \end{array}\right), \left(\begin{array}{c} 1^{(0)} \\ n^{(d-1)} \\ 2^{(d-1)} \\ \ldots \hspace{1cm} (n-1)^{(d-1)} \end{array}\right) \right]_T
  \]
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_2 : R_2^{(d-1)} \rightarrow R_n^{(d-1)}, \quad x \mapsto \left( \begin{pmatrix} 2^{(0)} & n^{(0)} \end{pmatrix} \right) x$$

Lemma (!, 2015)

The interval $\left( f_2 \left( R_2^{(d-1)} \right), \leq_T \right)$ is isomorphic to $\mathcal{NC}_G(1,1,n-2)$. 
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  \[ f_2 : R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)}, n^{(0)}\right)\right)x \]
- define $D_2 = R_2^{(d-1)} \uplus f_2\left(R_2^{(d-1)}\right)$, and $\mathcal{D}_2 = (D_2, \leq_T)$
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  $f_2 : R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( \binom{2}{0} \binom{n}{0} \right) x \right)$
- define $D_2 = R_2^{(d-1)} \biguplus f_2 \left( R_2^{(d-1)} \right)$, and $D_2 = (D_2, \leq_T)$

Lemma (Mühle, 2015)

The poset $D_2$ is isomorphic to $2 \times NC_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n - 1$. 
A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- define $D = R_n^{(d-1)} \setminus \left( f_1\left( R_1^{(1)} \right) \cup f_2\left( R_2^{(d-1)} \right) \right)$, and $\mathcal{D} = (D, \leq_T)$

Lemma (lâ, 2015)

The poset $\mathcal{D}$ is isomorphic to $\bigcup_{i=3}^{n-1} \mathcal{NC}_G(1,1,i-2) \times \mathcal{NC}_G(1,1,n-i)$. Moreover, its minimal elements have length 2, and its maximal elements have length $n - 2$. 
The Main Result

**Theorem ( *, 2015)**

For \( d, n \geq 2 \) the lattice \( \mathcal{NC}_{G(d,d,n)} \) admits a symmetric chain decomposition. Consequently, it is Peck.
Example: \( d = 5, n = 3 \)
Example: $d = 5, n = 3$
Motivation

Symmetric Chain Decompositions

Noncrossing Partition Lattices
  - Complex Reflection Groups
  - Noncrossing Partitions

Symmetric Chain Decompositions of $\mathcal{NC}_G(d,d,n)$
  - The Group $G(d,d,n)$
  - A First Decomposition
  - A Second Decomposition

Strong Sperner Property of $\mathcal{NC}_W$
The Remaining Cases

- so far: $\mathcal{NC}_G(1,1,n)$ and $\mathcal{NC}_G(d,d,n)$ admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?
The Remaining Cases

- so far: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$ admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?

**Theorem (V. Reiner, 1997)**

The lattice $\mathcal{NC}_{G(2,1,n)}$ admits a symmetric chain decomposition for any $n \geq 1$. 
The Remaining Cases

- so far: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$ admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?
- we have $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$ for $d \geq 2$ and $n \geq 1$

Theorem (V. Reiner, 1997)

The lattice $\mathcal{NC}_{G(2,1,n)}$ admits a symmetric chain decomposition for any $n \geq 1$. 
The Remaining Cases

- so far: $\mathcal{NC}_G(1,1,n)$ and $\mathcal{NC}_G(d,d,n)$ admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?
- we have $\mathcal{NC}_G(2,1,n) \cong \mathcal{NC}_G(d,1,n)$ for $d \geq 2$ and $n \geq 1$
- only the 26 exceptional groups remain

**Theorem (V. Reiner, 1997)**

The lattice $\mathcal{NC}_G(2,1,n)$ admits a symmetric chain decomposition for any $n \geq 1$. 
A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank $n$
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with $i$ largest ranks removed
A Decomposition Argument

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A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank $n$
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with $i$ largest ranks removed

\[ \mathcal{P}[1] \]
A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank $n$
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with $i$ largest ranks removed

Diagram of $\mathcal{P}[1]$: A hexagon with directed edges connecting the vertices.
A Decomposition Argument

- \( \mathcal{P} \) .. graded poset of rank \( n \)
- \( \mathcal{P}[i] \) .. subposet of \( \mathcal{P} \) with \( i \) largest ranks removed
A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank $n$
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with $i$ largest ranks removed
A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank $n$
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with $i$ largest ranks removed

$\mathcal{P}[3]$
Motivation
Symmetric Chain Decompositions
NCP
Complex Reflection Groups
Noncrossing Partitions

SCD of $\mathcal{NC}_G(d,d,n)$
The Group $G(d,d,n)$
A First Decomposition
A Second Decomposition

SSP of $\mathcal{NC}_W$

A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank $n$
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with $i$ largest ranks removed

$\mathcal{P}[3]$
A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank $n$
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with $i$ largest ranks removed

$\mathcal{P}[4]$
A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank $n$
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with $i$ largest ranks removed

Proposition (ซะ, 2015)

A graded poset $\mathcal{P}$ of rank $n$ is strongly Sperner if and only if $\mathcal{P}[i]$ is Sperner for all $i \in \{0, 1, \ldots, n\}$.

- antichains in $\mathcal{P}[i]$ are antichains in $\mathcal{P}[s]$ for $s < i$
A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank $n$
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with $i$ largest ranks removed

Proposition (חש, 2015)

A graded poset $\mathcal{P}$ of rank $n$ is strongly Sperner if and only if $\mathcal{P}[i]$ is Sperner for all $i \in \{0, 1, \ldots, n\}$.

- antichains in $\mathcal{P}[i]$ are antichains in $\mathcal{P}[s]$ for $s < i$
A Decomposition Argument

- SAGE has a fast implementation to compute the size of the largest antichain of a poset
A Decomposition Argument

- **SAGE** has a fast implementation to compute the *width* of a poset
**A Decomposition Argument**

- **SAGE** has a fast implementation to compute the **width** of a poset

**Theorem (_mpi, 2015)**

*The lattice $\mathcal{NC}_W$ is Peck for any well-generated exceptional complex reflection group $W$.***
A Decomposition Argument

- **SAGE** has a fast implementation to compute the **width** of a poset

### Theorem (†, 2015)

The lattice $\mathcal{NC}_W$ is Peck for any well-generated complex reflection group group $W$. 

$m$-Divisible Noncrossing Partition Posets

- $W$.. well-generated complex reflection group; $c$.. Coxeter element of $W$
- $m$-divisible noncrossing partition: $m$-multichain of noncrossing partitions

$$(w)_m = (w_1, w_2, \ldots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \cdots \leq_T w_m \leq_T c$$
m-Divisible Noncrossing Partition Posets

- \(W\) .. well-generated complex reflection group; \(c\) .. Coxeter element of \(W\)
- \textbf{\(m\)-divisible noncrossing partition}: \(m\)-multichain of noncrossing partitions \(\sim \text{NC}_W^{(m)}(c)\)
- \textbf{\(m\)-delta sequence}: sequence of “differences” of elements in a multichain

\[ (w)_m = (w_1, w_2, \ldots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \cdots \leq_T w_m \leq_T c \]
\[ \partial(w)_m = [w_1; w_1^{-1}w_2, w_2^{-1}w_3, \ldots, w_{m-1}^{-1}w_m, w_m^{-1}c] \]
**m-Divisible Noncrossing Partition Posets**

- \( W \): well-generated complex reflection group; \( c \): Coxeter element of \( W \)
- **\( m \)-divisible noncrossing partition**: \( m \)-multichain of noncrossing partitions
- **\( m \)-delta sequence**: sequence of “differences” of elements in a multichain
- partial order: \( (u)_m \leq (v)_m \) if and only if \( \partial(u)_m \leq_T \partial(v)_m \)

**Question (D. Armstrong, 2009)**

Are the posets \( \mathcal{NC}_W^{(m)} \) strongly Sperner for any \( W \) and any \( m \geq 1 \)?
Motivation
Symmetric Chain Decompositions
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Noncrossing Partitions

SCD of $\mathcal{NC}_G(d,d,n)$

$\mathcal{NC}_G(d,d,n)$

A First Decomposition
A Second Decomposition

SSP of $\mathcal{NC}_W$

$m$-Divisible Noncrossing Partition Posets

- affirmative answer for $m = 1$

Question (D. Armstrong, 2009)

Are the posets $\mathcal{NC}_W^{(m)}$ strongly Sperner for any $W$ and any $m \geq 1$?


$m$-Divisible Noncrossing Partition Posets

- affirmative answer for $m = 1$
- what about $m > 1$?
  - $\mathcal{NC}_W^{(m)}$ is antiisomorphic to an order ideal in $(\mathcal{NC}_W)^m$
  - $(\mathcal{NC}_W)^m$ is Peck
  - $\mathcal{NC}_W^{(m)}$ is not rank-symmetric $\leadsto$ no symmetric chain decomposition

Question (D. Armstrong, 2009)

Are the posets $\mathcal{NC}_W^{(m)}$ strongly Sperner for any $W$ and any $m \geq 1$?
Example: $\mathcal{NC}_{\mathfrak{S}_4}^{(2)}$
Example: $\mathcal{NC}_G^{(2)}$
Thank You.
Interlude: A Convolution Formula

\[ \mathcal{N}_G^{(d,d,n)}(c) = R_1^{(0)} \uplus R_1^{(1)} \uplus \biguplus_{i=2}^{n-1} \left( R_i^{(0)} \uplus R_i^{(d-1)} \right) \uplus \biguplus_{s=0}^{d-1} R_n^{(s)} \]
Proposition (𝘍, 2015)

For $n \geq 0$ we have

$$\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n+1}{n-1}.$$ 

$$\mathcal{NC}_{G(d,d,n)}(c) = R_1^{(0)} \uplus R_1^{(1)} \uplus \bigcup_{i=2}^{n-1} \left( R_i^{(0)} \uplus R_i^{(d-1)} \right) \uplus \bigcup_{s=0}^{d-1} R_n^{(s)}.$$
Proposition ( ListBox, 2015)

For $n \geq 0$ we have \[
\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n + 1}{n - 1}.
\]

\[
\text{Cat}_{G(d,d,n+2)} = 2 \cdot \text{Cat}_{G(d,d,n+1)} + 2 \cdot \text{Cat}_{G(1,1,n)} + d \cdot \text{Cat}_{G(1,1,n+1)}
+ 2 \sum_{i=3}^{n+1} \text{Cat}_{G(d,d,n-i+3)} \text{Cat}_{G(1,1,i-2)}
\]
Proposition (❄, 2015)

For $n \geq 0$ we have

$$\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n+1}{n-1}.$$
Proposition (_RDONLY, 2015)

For \( n \geq 0 \) we have

\[
\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n+1}{n-1}.
\]

\[
\text{Cat}_{G(d,d,n+2)} = d \cdot \text{Cat}_{G(1,1,n+1)}
\]

\[
+ 2 \cdot \sum_{i=0}^{n} \text{Cat}_{G(d,d,n-i+1)} \cdot \text{Cat}_{G(1,1,i)}
\]
Proposition (**, 2015)

For $n \geq 0$ we have

$$\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n + 1}{n - 1}.$$ 

$$\text{Cat}_{G(d,d,n+2)} = d \cdot \text{Cat}_{G(1,1,n+1)}$$

$$+ 2 \cdot \sum_{i=0}^{n} \text{Cat}_{G(d,d,i+1)} \cdot \text{Cat}_{G(1,1,n-i)}$$
Proposition (Moon, 2015)

For \( n \geq 0 \) we have

\[
\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n + 1}{n-1}.
\]

\[
\text{Cat}_{G(d,d,n+2)} = d \cdot \text{Cat}_{G(1,1,n+1)} + 2 \cdot \sum_{i=0}^{n} \text{Cat}_{G(d,d,i+1)} \cdot \text{Cat}_{G(1,1,n-i)}
\]

\[
\text{Cat}_{G(d,d,n+2)} = \left( \prod_{i=1}^{n+1} \frac{di + (n - 1)d}{di} \right) \frac{n + (n - 1)d}{n}
\]
Proposition (∗, 2015)

For $n \geq 0$ we have

$$\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n+1}{n-1}.$$

\[
\text{Cat}_{G(d,d,n+2)} = d \cdot \text{Cat}_{G(1,1,n+1)} \\
+ 2 \cdot \sum_{i=0}^{n} \text{Cat}_{G(d,d,i+1)} \cdot \text{Cat}_{G(1,1,n-i)} \\
\text{Cat}_{G(d,d,n+2)} = \left( (n+1)d + n + 2 \right) \cdot \text{Cat}_{G(1,1,n+1)}
\]
Interlude: A Convolution Formula

Proposition (†, 2015)

For $n \geq 0$ we have

$$\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n + 1}{n - 1}.$$ 

\[
\text{Cat}_{G(d,d,n+2)} = d \cdot \text{Cat}_{G(1,1,n+1)} \\
+ 2 \cdot \sum_{i=0}^{n} \text{Cat}_{G(d,d,i+1)} \cdot \text{Cat}_{G(1,1,n-i)}
\]

\[
\text{Cat}_{G(d,d,n+2)} = d \cdot \text{Cat}_{G(1,1,n+1)} \\
+ \left( nd + n + 2 \right) \cdot \text{Cat}_{G(1,1,n+1)}
\]
Proposition (§, 2015)

For $n \geq 0$ we have
$$\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n + 1}{n - 1}.$$
Proposition (араметр, 2015)

For $n \geq 0$ we have

$$\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n + 1}{n - 1}.$$ 

$$nd \cdot \text{Cat}_{G(1,1,n+1)} + \binom{2(n + 1)}{n + 1} =$$

$$2 \cdot \sum_{i=0}^{n} \left( id \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} \right) + 2 \cdot \sum_{i=0}^{n} \binom{2i}{i} \cdot \text{Cat}_{G(1,1,n-i)}$$
Proposition (❄, 2015)

For $n \geq 0$ we have

$$\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n+1}{n-1}.$$ 

$$nd \cdot \text{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} =$$

$$2d \cdot \sum_{i=0}^{n} \left( i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} \right) + \binom{2(n+1)}{n+1}.$$
Interlude: A Convolution Formula

Proposition (®, 2015)

For $n \geq 0$ we have
\[
\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n+1}{n-1}.
\]

\[
\frac{n}{2} \cdot \text{Cat}_{G(1,1,n+1)} = \sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)}
\]
Proposition (📸, 2015)

For \( n \geq 0 \) we have

\[
\sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)} = \binom{2n + 1}{n - 1}.
\]

\[
\binom{2n + 1}{n - 1} = \sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)}
\]
Proposition (Y. Kong, 2000)

For \( n \geq 0 \) we have

\[
\sum_{i=0}^{n-1} \text{Cat}_{G(1,1,i)} \cdot \binom{2(n-i)}{n-i-1} = \binom{2n+1}{n-1}.
\]