On the EL-Shellability of the Cambrian Lattices

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Motivation

- it is well-known that the Hasse diagram of the Tamari lattice corresponds to the 1-skeleton of the classical associahedron
- the Tamari lattice $T_n$ can be realized as a lattice quotient of the weak order lattice of the Coxeter group $A_n$
- the bottom elements of each congruence class are precisely the 312-avoiding permutations
- Nathan Reading has generalized this construction to all finite Coxeter groups $W$ and all Coxeter elements $\gamma \in W$
- he called the resulting lattices *Cambrian lattices*, denoted by $C_\gamma$
- this construction yields a generalized associahedron for all finite Coxeter groups
Motivation

• Björner and Wachs showed that $T_n$ is EL-shellable and that each open interval of $T_n$ is either contractible or spherical

• it follows from a result by Nathan Reading that the open intervals of $C_\gamma$ are either contractible or spherical
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  • it follows from a result by Hugh Thomas and Colin Ingalls that $C_\gamma$ is EL-shellable
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- however,
  - Thomas and Ingalls utilize the representation theory of Coxeter groups
  - Reading utilizes the fact that $C_\gamma$ is the fan lattice of the Coxeter arrangement
Motivation

- Björner and Wachs showed that $\mathcal{T}_n$ is EL-shellable and that each open interval of $\mathcal{T}_n$ is either contractible or spherical
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- however,
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- we give a direct, case-free proof of these properties, using the realization of $C_\gamma$ in terms of $\gamma$-sortable elements
Outline

1 Preliminaries
   Cambrian Lattices
   EL-Shellability of Posets

2 EL-Shellability of $C_\gamma$
   The Labeling
   Main Result

3 Applications
   Topology of $C_\gamma$
   Subword Complexes
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1 Preliminaries
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\(\gamma\)-Sorting Words

- Let \(W\) be a finite Coxeter group of rank \(n\), with simple generators \(S = \{s_1, s_2, \ldots, s_n\}\)
- Consider the Coxeter element \(\gamma = s_1 s_2 \cdots s_n\) and the half-infinite word \(\gamma^\infty = s_1 s_2 \cdots s_n|s_1 s_2 \cdots s_n|s_1 \cdots\)
- \(\gamma\)-sorting word of \(w\): the reduced decomposition of \(w \in W\) which is lexicographically first as a subword of \(\gamma^\infty\) among all reduced decompositions of \(w\)
\( \gamma \)-Sorting Words – Example

- Let \( W = A_4 \) with \( s_i = (i, i + 1) \), and \( \gamma = s_1 s_2 s_3 s_4 \)
- Consider \( w = s_1 s_4 s_3 s_4 \)
- There are eight reduced decompositions of \( w \), namely
  \[
  s_1 s_4 s_3 s_4, \quad s_4 s_1 s_3 s_4, \quad s_4 s_3 s_1 s_4, \quad s_4 s_3 s_4 s_1, \\
  s_1 s_3 s_4 s_3, \quad s_3 s_1 s_4 s_3, \quad s_3 s_4 s_1 s_3, \quad s_3 s_4 s_3 s_1
  \]
- The decomposition \( s_1 s_3 s_4 s_3 \) is the lexicographically first subword of \( \gamma^\infty \) among these words, and thus the \( \gamma \)-sorting word of \( w \)
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  \[ s_1 s_4 s_3 s_4, \quad s_4 s_1 s_3 s_4, \quad s_4 s_3 s_1 s_4, \quad s_4 s_3 s_4 s_1, \]
  \[ s_1 s_3 s_4 s_3, \quad s_3 s_1 s_4 s_3, \quad s_3 s_4 s_1 s_3, \quad s_3 s_4 s_3 s_1 \]
- the decomposition \( s_1 s_3 s_4 s_3 \) is the lexicographically first subword of \( \gamma^\infty \) among these words, and thus the \( \gamma \)-sorting word of \( w \)

\[ S_1 S_2 S_3 S_4 \mid S_1 S_2 S_3 S_4 \mid S_1 S_2 S_3 S_4 \]
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- let \( W = A_4 \) with \( s_i = (i, i + 1) \), and \( \gamma = s_1s_2s_3s_4 \)
- consider \( w = s_1s_4s_3s_4 \)
- there are eight reduced decompositions of \( w \), namely
  \[ s_1s_4s_3s_4, \quad s_4s_1s_3s_4, \quad s_4s_3s_1s_4, \quad s_4s_3s_4s_1, \]
  \[ s_1s_3s_4s_3, \quad s_3s_1s_4s_3, \quad s_3s_4s_1s_3, \quad s_3s_4s_3s_1 \]
- the decomposition \( s_1s_3s_4s_3 \) is the lexicographically first subword of \( \gamma^\infty \) among these words, and thus the \( \gamma \)-sorting word of \( w \)

\[ s_1s_2s_3s_4|s_1s_2s_3s_4|s_1s_2s_3s_4 \]
\(\gamma\text{-Sorting Words – Example}\)

- let \(W = A_4\) with \(s_i = (i, i + 1)\), and \(\gamma = s_1s_2s_3s_4\)
- consider \(w = s_1s_4s_3s_4\)
- there are eight reduced decompositions of \(w\), namely
  \[s_1s_4s_3s_4, \quad s_4s_1s_3s_4, \quad s_4s_3s_1s_4, \quad s_4s_3s_4s_1, \]
  \[s_1s_3s_4s_3, \quad s_3s_1s_4s_3, \quad s_3s_4s_1s_3, \quad s_3s_4s_3s_1\]
- the decomposition \(s_1s_3s_4s_3\) is the lexicographically first subword of \(\gamma^\infty\) among these words, and thus the \(\gamma\text{-sorting word of } w\)

\[s_1s_2s_3s_4 | s_1s_2s_3s_4 | s_1s_2s_3s_4\]
Let $W = A_4$ with $s_i = (i, i + 1)$, and $\gamma = s_1s_2s_3s_4$.

Consider $w = s_1s_4s_3s_4$.

There are eight reduced decompositions of $w$, namely:

$s_1s_4s_3s_4$, $s_4s_1s_3s_4$, $s_4s_3s_1s_4$, $s_4s_3s_4s_1$,

$s_1s_3s_4s_3$, $s_3s_1s_4s_3$, $s_3s_4s_1s_3$, $s_3s_4s_3s_1$.

The decomposition $s_1s_3s_4s_3$ is the lexicographically first subword of $\gamma^\infty$ among these words, and thus the $\gamma$-sorting word of $w$:

$\underline{s_1s_2s_3s_4 | s_1s_2s_3s_4 | s_1s_2s_3s_4}$
\textbf{γ-Sorting Words – Example}

- let $W = A_4$ with $s_i = (i, i+1)$, and $\gamma = s_1 s_2 s_3 s_4$
- consider $w = s_1 s_4 s_3 s_4$
- there are eight reduced decompositions of $w$, namely

  $s_1 s_4 s_3 s_4$, $s_4 s_1 s_3 s_4$, $s_4 s_3 s_1 s_4$, $s_4 s_3 s_4 s_1$,
  $s_1 s_3 s_4 s_3$, $s_3 s_1 s_4 s_3$, $s_3 s_4 s_1 s_3$, $s_3 s_4 s_3 s_1$

- the decomposition $s_1 s_3 s_4 s_3$ is the lexicographically first subword of $\gamma^\infty$ among these words, and thus the $\gamma$-sorting word of $w$

\[ s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 \]
\(\gamma\)-Sorting Words – Example

- let \(W = A_4\) with \(s_i = (i, i + 1)\), and \(\gamma = s_1s_2s_3s_4\)
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  \[s_1s_4s_3s_4, \quad s_4s_1s_3s_4, \quad s_4s_3s_1s_4, \quad s_4s_3s_4s_1,\]
  \[s_1s_3s_4s_3, \quad s_3s_1s_4s_3, \quad s_3s_4s_1s_3, \quad s_3s_4s_3s_1\]
- the decomposition \(s_1s_3s_4s_3\) is the lexicographically first subword of \(\gamma^{\infty}\) among these words, and thus the \(\gamma\)-sorting word of \(w\)

\[s_1s_2s_3s_4 | s_1s_2s_3s_4 | s_1s_2s_3s_4\]
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  \]
- the decomposition \(s_1s_3s_4s_3\) is the lexicographically first subword of \(\gamma^\infty\) among these words, and thus the \(\gamma\)-sorting word of \(w\)

\[
  s_1s_2s_3s_4\,|\,s_1s_2s_3s_4\,|\,s_1s_2s_3s_4
  
\]
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- let \(W = A_4\) with \(s_i = (i, i+1)\), and \(\gamma = s_1 s_2 s_3 s_4\)
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- there are eight reduced decompositions of \(w\), namely
  \[
  s_1 s_4 s_3 s_4, \quad s_4 s_1 s_3 s_4, \quad s_4 s_3 s_1 s_4, \quad s_4 s_3 s_4 s_1,
  \]
  \[
  s_1 s_3 s_4 s_3, \quad s_3 s_1 s_4 s_3, \quad s_3 s_4 s_1 s_3, \quad s_3 s_4 s_3 s_1
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\[
S_1 S_2 S_3 S_4 | S_1 S_2 S_3 S_4 | S_1 S_2 S_3 S_4
\]
\(\gamma\)-Sorting Words – Example

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  s_1 s_3 s_4 s_3, \quad s_3 s_1 s_4 s_3, \quad s_3 s_4 s_1 s_3, \quad s_3 s_4 s_3 s_1
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S_1 S_2 S_3 S_4 | S_1 S_2 S_3 S_4 | S_1 S_2 S_3 S_4
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- The decomposition \( s_1 s_3 s_4 s_3 \) is the lexicographically first subword of \( \gamma^\infty \) among these words, and thus the \( \gamma \)-sorting word of \( w \)

\[
S_1 S_2 S_3 S_4 \mid S_1 S_2 S_3 S_4 \mid S_1 S_2 S_3 S_4
\]
\(\gamma\)-Sortable Words

- write the \(\gamma\)-sorting word of \(w\) as follows

\[
\gamma\text{-sorting word } w = s_{1}^{\delta_{1,1}} s_{2}^{\delta_{1,2}} \cdots s_{n}^{\delta_{1,n}} | s_{1}^{\delta_{2,1}} s_{2}^{\delta_{2,2}} \cdots s_{n}^{\delta_{2,n}} | \cdots | s_{1}^{\delta_{l,1}} s_{2}^{\delta_{l,2}} \cdots s_{n}^{\delta_{l,n}},
\]

where \(\delta_{i,j} \in \{0, 1\}\) for \(1 \leq i \leq l\) and \(1 \leq j \leq n\)

- \(i\)-th block of \(w\): the set \(b_{i} = \{s_{j} \mid \delta_{i,j} = 1\} \subseteq S\), where \(i \in \{1, 2, \ldots, l\}\)

- \(\gamma\)-sortable word: a word \(w \in W\) satisfying \(b_{1} \supseteq b_{2} \supseteq \cdots \supseteq b_{l}\)
\( \gamma \)-Sortable Words

- write the \( \gamma \)-sorting word of \( w \) as follows

\[
w = s_{\delta_{1,1}} s_{\delta_{1,2}} \cdots s_{\delta_{1,n}} | s_{\delta_{2,1}} s_{\delta_{2,2}} \cdots s_{\delta_{2,n}} | \cdots | s_{\delta_{l,1}} s_{\delta_{l,2}} \cdots s_{\delta_{l,n}},
\]

where \( \delta_{i,j} \in \{0, 1\} \) for \( 1 \leq i \leq l \) and \( 1 \leq j \leq n \)

- \( i \)-th block of \( w \): the set \( b_i = \{ s_j \mid \delta_{i,j} = 1 \} \subseteq S \), where \( i \in \{1, 2, \ldots, l\} \)

- \( \gamma \)-sortable word: a word \( w \in W \) satisfying \( b_1 \supseteq b_2 \supseteq \cdots \supseteq b_l \)

- the \( \gamma \)-sorting word \( w = s_1 s_3 s_4 | s_3 \) has \( b_1 = \{ s_1, s_3, s_4 \} \) and \( b_2 = \{ s_3 \} \) and is thus \( \gamma \)-sortable

- the \( \gamma \)-sorting word \( v = s_1 s_3 s_4 | s_2 \) is not
Theorem (Reading, 2005)

Let $\gamma$ be a Coxeter element of a finite Coxeter group $W$. The $\gamma$-sortable elements of $W$ constitute a sublattice of the weak order on $W$.

- consider the map $\pi_\gamma : W \to W$, $w \mapsto \pi_\gamma(w)$ that maps $w$ to the largest $\gamma$-sortable element below it
- the fibers of $\pi_\gamma$ induce a lattice congruence $\theta_\gamma$ on the weak order on $W$
- Cambrian lattice $C_\gamma$: the lattice quotient $W/\theta_\gamma$
Cambrian Lattices – Example
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\[ \gamma = s_1 s_2 s_3 \]
Cambrian Lattices – Example

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\[ \gamma = s_3 s_2 s_1 \]
Cambrian Lattices – Example

\[ \gamma = s_2 s_1 s_3 \]
Basics on Posets

- **bounded poset**: a poset that has a unique minimal and a unique maximal element
- Let $\mathbb{P} = (P, \leq_{\mathbb{P}})$ be a bounded poset
- $\mathbb{P}$ is the poset that arises from $\mathbb{P}$ by removing the maximal and minimal element (the so-called proper part of $\mathbb{P}$)
- **chain**: linearly ordered subset $c$ of $P$
  - notation: $c : p_0 <_{\mathbb{P}} p_1 <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_s$
- **maximal chain in $[p, q]$**: there is no $p' \in [p, q]$ and no $0 \leq i < s$ such that
  - $p = p_0 <_{\mathbb{P}} p_1 <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_i <_{\mathbb{P}} p' <_{\mathbb{P}} p_{i+1} <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_s = q$
  - is a chain
Edge-Labelings

- cover relation $p <_P q$: $p <_P q$ and there is no $p' \in P$ with $p <_P p' <_P q$
- $\mathcal{E}(P) = \{(p, q) \mid p <_P q\}$ is the set of covering relations on $P$
- edge-labeling $\lambda$: map $\lambda : \mathcal{E}(P) \to \Lambda$, for some poset $\Lambda$
- $\lambda(c) = (\lambda(p_0, p_1), \lambda(p_1, p_2), \ldots, \lambda(p_{s-1}, p_s))$ is the label-sequence of $c$
- rising chain: a chain $c$ such that $\lambda(c)$ is strictly increasing
- ER-labeling: an edge-labeling such that for every interval of $P$ there is exactly one rising maximal chain
- EL-labeling: an ER-labeling such that the rising chain in every interval is lexicographically first among all maximal chains
EL-Shellability

- **EL-shellable poset**: a bounded poset that admits an EL-labeling
EL-Shellability

- **EL-shellable poset**: a bounded poset that admits an EL-labeling
- the order complex $\Delta(\mathcal{P})$ of an EL-shellable poset $\mathcal{P}$ is shellable and hence Cohen-Macaulay
- the geometric realization of $\Delta(\mathcal{P})$ is homotopy equivalent to a wedge of spheres
- the $i$-th Betti number of $\Delta(\mathcal{P})$ is given by the number of falling maximal chains of length $i + 2$
- hence, the Euler characteristic $\chi(\Delta(\mathcal{P}))$ can be computed from the labeling
- if $0_\mathcal{P}$ is the unique minimal element and $1_\mathcal{P}$ the unique maximal element of $\mathcal{P}$, we have $\chi(\Delta(\mathcal{P})) = \mu(0_\mathcal{P}, 1_\mathcal{P})$
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The Labeling

- recall that we write the $\gamma$-sorting word of $w \in W$ as
  \[ w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}}, \]
  where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
- define the set of filled positions of $w$ in $\gamma^\infty$ by
  \[ \alpha(w) = \{(i - 1) \cdot n + j \mid \delta_{i,j} = 1\} \subseteq \mathbb{N} \]
The Labeling

- recall that we write the $\gamma$-sorting word of $w \in W$ as
  $$w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}},$$
  where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$

- define the set of filled positions of $w$ in $\gamma^\infty$ by
  $$\alpha(w) = \{(i - 1) \cdot n + j | \delta_{i,j} = 1\} \subseteq \mathbb{N}$$

- let $w = s_1 s_3 | s_2 s_4 | s_3 \in A_4$
The Labeling

- recall that we write the $\gamma$-sorting word of $w \in \mathcal{W}$ as
  \[ w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}}, \]
  where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
- define the set of filled positions of $w$ in $\mathcal{L}_\infty$ by
  \[ \alpha(w) = \{(i - 1) \cdot n + j \mid \delta_{i,j} = 1\} \subseteq \mathbb{N} \]
  
- let $w = s_1 s_3 | s_2 s_4 | s_3 \in \mathcal{A}_4$

  $w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0,$
The Labeling

- recall that we write the $\gamma$-sorting word of $w \in W$ as
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  where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
- define the set of filled positions of $w$ in $\gamma^\infty$ by
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- let $w = s_1 s_3 | s_2 s_4 | s_3 \in A_4$
  \[ w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1\} \]
The Labeling

- recall that we write the $\gamma$-sorting word of $w \in \mathcal{W}$ as
  \[ w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}}, \]
  where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
- define the set of filled positions of $w$ in $\gamma^\infty$ by
  \[ \alpha(w) = \{(i - 1) \cdot n + j \mid \delta_{i,j} = 1\} \subseteq \mathbb{N} \]
- let $w = s_1 s_3 | s_2 s_4 | s_3 \in A_4$
  \[ w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1\} \]
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  where $\delta_{i,j} \in \{0,1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
- define the set of filled positions of $w$ in $\gamma^\infty$ by
  \[ \alpha(w) = \{(i - 1) \cdot n + j | \delta_{i,j} = 1\} \subseteq \mathbb{N} \]
- let $w = s_1 s_3 | s_2 s_4 | s_3 \in A_4$
  \[ \begin{align*}
    w &= s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \\
    \alpha(w) &= \{1, 3\}
  \end{align*} \]
The Labeling

• recall that we write the $\gamma$-sorting word of $w \in W$ as

$$w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}},$$

where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$

• define the set of filled positions of $w$ in $\gamma^\infty$ by

$$\alpha(w) = \{(i - 1) \cdot n + j \mid \delta_{i,j} = 1\} \subseteq \mathbb{N}$$

• let $w = s_1 s_3 | s_2 s_4 | s_3 \in A_4$

$$w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1, 3\}$$
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- recall that we write the \( \gamma \)-sorting word of \( w \in W \) as

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\]

where \( \delta_{i,j} \in \{0, 1\} \) for \( 1 \leq i \leq l \) and \( 1 \leq j \leq n \)

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\]

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\[
    w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1, 3\}
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  where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
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  \[ w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1, 3, 6\} \]
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where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$

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• let $w = s_1 s_3 | s_2 s_4 | s_3 \in A_4$

$$w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1, 3, 6\}$$
The Labeling

- recall that we write the $\gamma$-sorting word of $w \in \mathcal{W}$ as
  \[ w = s_{1,1}^1 s_2^1 \cdots s_n^1 | s_{1,1}^2 s_2^2 \cdots s_n^2 | \cdots | s_{1,1}^l s_2^l \cdots s_n^l, \]
  where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
- define the set of filled positions of $w$ in $\gamma^\infty$ by
  \[ \alpha(w) = \{(i - 1) \cdot n + j \mid \delta_{i,j} = 1\} \subseteq \mathbb{N} \]
- let $w = s_1 s_3 s_2 s_4 | s_3 s_2 s_4 s_1 \in A_4$
  \[ w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1, 3, 6, 8\} \]
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  \[ w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1, 3, 6, 8\} \]
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- let $w = s_1 s_3 | s_2 s_4 | s_3 \in A_4$
  
  \[ w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1, 3, 6, 8, 11\} \]
recall that we write the $\gamma$-sorting word of $w \in W$ as
\[ w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}}, \]
where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$

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\[ w = s_1^1 s_2^0 s_3^1 s_4^1 \big| s_1^0 s_2^1 s_3^0 s_4^1 \big| s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1, 3, 6, 8, 11\} \]
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- recall that we write the $\gamma$-sorting word of $w \in W$ as
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  where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
- define the set of filled positions of $w$ in $\gamma^\infty$ by
  \[ \alpha(w) = \{(i - 1) \cdot n + j \mid \delta_{i,j} = 1\} \subseteq \mathbb{N} \]
- $\lambda : \mathcal{E}(C_\gamma) \to \mathbb{N}, \ (u, v) \mapsto \min\{\alpha(v) \setminus \alpha(u)\}$
The Labeling – Example
Main Result

**Theorem**

For every finite Coxeter group $W$ and every Coxeter element $\gamma \in W$, the edge-labeling $\lambda$ is an EL-labeling of $C_\gamma$.

We need two technical lemmas for the proof!
Lemma 1

Let $u \leq v$ in $C_\gamma$. If $u$ and $v$ have the same first block $b$, then let $u', v'$ be the elements obtained by omitting $b$. Then, $u', v' \in C_\gamma$, and we have:

1. The intervals $[u, v]$ and $[u', v']$ are isomorphic.
2. For every $w'_1, w'_2 \in [u', v']$ with $w'_1 \preceq w'_2$ we have
   $\lambda(bw'_1, bw'_2) = \lambda(w'_1, w'_2) + n$. 
Lemma 2

Lemma

For $u, v \in C_\gamma$ with $u \leq v$ define $i_0 = \min \{ i \in \alpha(v) \setminus \alpha(u) \}$. The following hold:

1. The label $i_0$ appears in every maximal chain of $[u, v]$.
2. There is a unique element $u_1 \in (u, v)$ with $u \preceq u_1$ and $\lambda(u, u_1) = i_0$.
3. $\alpha(u)$ is a subset of $\alpha(v)$.
4. The labels of each maximal chain in $[u, v]$ are distinct.
Main Result

**Theorem**

For every finite Coxeter group $W$ and every Coxeter element $\gamma \in W$, the edge-labeling $\lambda$ is an EL-labeling of $C_\gamma$.

**Sketch of proof:**

- proceed by induction on the length $k$ of the interval $[u, v]$
- if $k = 2$, then the result follows from Lemma 2
- Lemma 2 tells us that there exists an $u \lessdot u_1$ in $[u, v]$ with $\lambda(u, u_1) = i_0$
- apply induction on the interval $[u_1, v]$ to find the maximal chain $u_1 \lessdot u_2 \lessdot \cdots \lessdot v$ which is rising and lexicographically first
- by definition and Lemma 2, the chain $u \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot v$ is the desired maximal chain in $[u, v]$
Outline

1 Preliminaries
   Cambrian Lattices
   EL-Shellability of Posets

2 EL-Shellability of $C_\gamma$
   The Labeling
   Main Result

3 Applications
   Topology of $C_\gamma$
   Subword Complexes
Theorem (Reading, 2004)

Every open interval in a Cambrian lattice is either contractible or homotopy equivalent to a sphere.

- Nathan Reading obtained this result by showing that \( C_\gamma \) is a special instance of a fan lattice associated to a central hyperplane arrangement
- he showed this property for this larger class of lattices
- having an EL-labeling of \( C_\gamma \), we can proof this property directly
Topology of $C_\gamma$

**Theorem**

Let $u, v \in C_\gamma$ with $u \leq v$. Then $|\mu(u, v)| \leq 1$.

- if $\mathcal{P}$ is an EL-shellable poset, and $p, q \in \mathcal{P}$ with $p \leq q$, then
  
  $\mu(p, q) = \# \text{ even length falling chains in } [p, q] - \# \text{ odd length falling chains in } [p, q]$

- we show that there exists at most one falling chain in each interval
Subword Complexes

- Vincent Pilaud and Christian Stump have recently shown that the Cambrian lattices coincide with the poset of flips of special subword complexes.
- Christian Stump observed that our labeling is a specialization of a natural labeling of the poset of flips for every subword complex.
Thank You.
An EL-Labeling for Trim Lattices

• let $L$ be a lattice
• left-modular element: $x \in L$ such that for all $y, z \in L$ holds
  $$(y \lor_L x) \land_L z = y \lor_L (x \land_L z)$$
• left-modular lattice: a lattice that contains a maximal chain of left-modular elements
• join-irreducible element: $x \in L$ which covers exactly one element
• meet-irreducible element: $x \in L$ which is covered by exactly one element
• trim lattice: a left-modular lattice (with left-modular chain of length $n$) that has exactly $n$ join- and $n$ meet-irreducible elements
An EL-Labeling for Trim Lattices

- let $L$ be a finite lattice with left-modular chain
  $\hat{0} = x_0 \leq_L x_1 \leq_L \cdots \leq_L x_n = \hat{1}$
- $\gamma : \mathcal{E}(L) \to \mathbb{N}, \ (p, q) \mapsto \min \{ i \mid p \lor_L x_i \land_L q = q \}$

Proposition (Liu, 1999)

If $L$ is a finite, left-modular lattice, then $\gamma$ is an EL-labeling.
Liu’s Labeling
Our Labeling