EL-Shellability of the $m$-Tamari Lattices

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Origin

- the $m$-Tamari lattices were introduced by Bergeron and Préville-Ratelle in order to express the Frobenius characteristics of the space of higher diagonal harmonics
- Bousquet-Mélou, Fusy and Préville-Ratelle proved the lattice property and a formula for the number of intervals
- combinatorial realization via $m$-Dyck paths
the $m$-Tamari lattices were introduced by Bergeron and Prévile-Ratelle in order to express the Frobenius characteristics of the space of higher diagonal harmonics.

Bousquet-Mélou, Fusy and Prévile-Ratelle proved the lattice property and a formula for the number of intervals.

Combinatorial realization via $m$-Dyck paths.

We are interested in topological properties of $\mathcal{T}_n^{(m)}$, which can be determined with the help of EL-shellability.
Outline

1 Preliminaries
   - $m$-Tamari Lattices
   - EL-Shellability of Posets

2 EL-Shellability of $\mathcal{T}_n^{(m)}$
   - A natural Edge-Labeling
   - Constructing Rising Chains

3 Topological Properties of $\mathcal{T}_n^{(m)}$
   - Falling Maximal Chains
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   - Falling Maximal Chains
**m-Dyck Paths**

- **m-Dyck path**: lattice path in $\mathbb{Z}^2$ from $(0, 0)$ to $(mn, n)$ that stays above the line $y = mx$
- only up-steps and right-steps are allowed
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- only up-steps and right-steps are allowed
- a 4-Dyck path of height 6
**$m$-Dyck Paths**

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- only up-steps and right-steps are allowed
- *not* a 4-Dyck path
\( m \)-Dyck Paths

- \( \mathcal{D}^{(m)}_n \): set of \( m \)-Dyck paths of height \( n \)
- associate an integer sequence \( \alpha_p = (a_1, a_2, \ldots, a_n) \) to \( p \in \mathcal{D}^{(m)}_n \) that satisfies

\[
a_1 \leq a_2 \leq \cdots \leq a_n, \quad \text{and} \\
a_i \leq m(i - 1), \quad 1 \leq i \leq n
\]
$m$-Dyck Paths

- $D_n^{(m)}$: set of $m$-Dyck paths of height $n$
- associate an integer sequence $\alpha_p = (a_1, a_2, \ldots, a_n)$ to $p \in D_n^{(m)}$ that satisfies

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![Diagram of $m$-Dyck paths](image-url)
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$m$-Dyck Paths

- **$m$-Dyck subpath at position $i$**: the unique subpath of $p$ that begins at the $i$-th upstep of $p$ and is an $m$-Dyck path again
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\[ a_6 = 15, \quad a_5 = 4, \quad a_4 = 4, \quad a_3 = 3, \quad a_2 = 0, \quad a_1 = 0 \]
$m$-Dyck Paths

- $m$-Dyck subpath at position $i$: the unique subpath of $p$ that begins at the $i$-th upstep of $p$ and is an $m$-Dyck path again

Graph showing $m$-Dyck paths with labels $a_1 = 0$, $a_2 = 0$, $a_3 = 3$, $a_4 = 4$, $a_5 = 4$, $a_6 = 15$.
**m-Dyck Paths**

- **m-Dyck subpath at position** $i$: the unique subpath of $p$ that begins at the $i$-th upstep of $p$ and is an $m$-Dyck path again

\[a_6 = 15, a_5 = 4, a_4 = 4, a_3 = 3, a_2 = 0, a_1 = 0\]

- **Primitive subsequence at position** $i$: unique subsequence $(a_i, a_{i+1}, \ldots, a_k)$ of $\alpha_p$ that satisfies

\[a_j - a_i < m(j - i), \quad i < j \leq k, \text{ and}\]

either $k = n$ or $a_{k+1} - a_i \geq m(k + 1 - i)$
A Covering Relation on $D_n^{(m)}$

- let $p \in D_n^{(m)}$, let $u$ be an upstep of $p$ that is preceded by a rightstep $r$
- say $u$ is the $i$-th upstep of $p$, and let $p_i$ be the $m$-Dyck subpath of $p$ at position $i$
- define $p \lessdot q$ if and only if $q$ is obtained from $p$ by exchanging $r$ and $p_i$
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- say $u$ is the $i$-th upstep of $p$, and let $p_i$ be the $m$-Dyck subpath of $p$ at position $i$
- define $p \preceq q$ if and only if $q$ is obtained from $p$ by exchanging $r$ and $p_i$

(0, 0, 3, 4, 4, 15) \preceq \\n\n(0, 0, 2, 3, 3, 15)
The \( m \)-Tamari Lattice

- let \( \leq \) denote the transitive and reflexive closure of \( \prec \)
- \textbf{\( m \)-Tamari lattice}: \( \mathcal{T}_n^{(m)} = (\mathcal{D}_n^{(m)}, \leq) \)
The $m$-Tamari Lattice

- let $\leq$ denote the transitive and reflexive closure of $\prec$
- **$m$-Tamari lattice**: $\mathcal{T}_n^{(m)} = (\mathcal{D}_n^{(m)}, \leq)$

- this is $\mathcal{T}_3^{(2)}$
The Main Question

Theorem (Björner & Wachs, 1997)

There exists an EL-labeling for $\mathcal{T}_n^{(1)}$ such that each interval has at most one falling chain.
The Main Question

Theorem (Björner & Wachs, 1997)

There exists an EL-labeling for $\mathcal{T}_n^{(1)}$ such that each interval has at most one falling chain.

- Can this result be generalized to $\mathcal{T}_n^{(m)}$ for $m \geq 1$?
Basics on Posets

- **bounded poset**: a poset that has a unique minimal and a unique maximal element
- let $\mathcal{P} = (P, \leq_P)$ be a bounded poset
- $\overline{\mathcal{P}}$ is the poset that arises from $\mathcal{P}$ by removing the maximal and minimal element (the so-called *proper part of* $\mathcal{P}$)
- **chain**: linearly ordered subset $c$ of $P$
  notation: $c : p_0 <_P p_1 <_P \cdots <_P p_s$
- **maximal chain in** $[p, q]$: there is no $p' \in [p, q]$ and no $0 \leq i < s$ such that
  $p = p_0 <_P p_1 <_P \cdots <_P p_i <_P p' <_P p_{i+1} <_P \cdots <_P p_s = q$
  is a chain
Edge-Labelings

- **cover relation** $p \prec_P q$: $p \prec_P q$ and there is no $p' \in P$ with $p \prec_P p' \prec_P q$
- $\mathcal{E}(P) = \{(p, q) \mid p \prec_P q\}$ is the set of covering relations on $P$
- **edge-labeling** $\lambda$: map $\lambda : \mathcal{E}(P) \to \Lambda$, for some poset $\Lambda$
- $\lambda(c) = (\lambda(p_0, p_1), \lambda(p_1, p_2), \ldots, \lambda(p_{s-1}, p_s))$ is the label-sequence of $c$
- **rising chain**: a chain $c$ such that $\lambda(c)$ is strictly increasing
- **ER-labeling**: an edge-labeling such that for every interval of $P$ there is exactly one rising maximal chain
- **EL-labeling**: an ER-labeling such that the rising chain in every interval is lexicographically first among all maximal chains
EL-Shellability

- **EL-shellable poset**: a bounded poset that admits an EL-labeling
EL-Shellability

- **EL-shellable poset**: a bounded poset that admits an EL-labeling

- the order complex $\Delta(P)$ of an EL-shellable poset $P$ is shellable and hence Cohen-Macaulay

- the geometric realization of $\Delta(P)$ is homotopy equivalent to a wedge of spheres

- the $i$-th Betti number of $\Delta(P)$ is given by the number of falling maximal chains of length $i + 2$

- hence, the Euler characteristic $\chi(\Delta(P))$ can be computed from the labeling

- if $0_P$ is the unique minimal element and $1_P$ the unique maximal element of $P$, we have $\chi(\Delta(P)) = \mu(0_P, 1_P)$
Möbius Function and Falling Chains
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2. EL-Shellability of $T_n^{(m)}$
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3. Topological Properties of $T_n^{(m)}$
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An Edge-Labeling

- an edge \((p, p')\) in \(\mathcal{T}_n^{(m)}\) is determined by the two sequences \(\alpha_p\) and \(\alpha_{p'}\), which satisfy

\[
\alpha_p = \alpha_{p'} + (0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0)
\]

\[
\underbrace{\phantom{0,0,\ldots,0}}_{i-1} \underbrace{\phantom{1,1,\ldots,1}}_{k-i+1} \underbrace{\phantom{0,0,\ldots,0}}_{n-k}
\]

- the value \(k\) is uniquely determined by \(i\)
- given \(\alpha_p = (a_1, a_2, \ldots, a_n)\) and \(i\), we can uniquely determine \(\alpha_{p'}\), and hence the covering pair \((p, p')\)
however, the position $i$ is not enough to distinguish the edges properly.

\begin{itemize}
  \item[(0, 0, 0)]
  \item[(0, 0, 1)]
  \item[(0, 0, 2)]
  \item[(0, 0, 3)]
  \item[(0, 0, 4)]
  \item[(0, 1, 2)]
  \item[(0, 1, 3)]
  \item[(0, 1, 4)]
  \item[(0, 2, 2)]
  \item[(0, 2, 3)]
  \item[(0, 2, 4)]
\end{itemize}
An Edge-Labeling

- to overcome this, we also take the value $a_i$ into account and consider the edge-labeling

$$\lambda : \mathcal{E}(\mathcal{T}_n^{(m)}) \to \mathbb{N} \times \mathbb{N}$$

$$(p, p') \mapsto (i, a_i),$$

where $\alpha_p = (a_1, a_2, \ldots, a_n)$ and

$$\alpha_p = \alpha_{p'} + (0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0)$$

- we consider the following linear order on the set of edge-labels

$$(i, a_i) < (j, a_j) \quad \text{if and only if} \quad i < j \quad \text{or} \quad i = j \text{ and } a_i > a_j$$
An Edge-Labeling
Constructing Rising Chains

- let $\alpha_p = (0, 0, 3, 4, 4, 15)$ and $\alpha_q = (0, 0, 1, 1, 1, 13)$
Constructing Rising Chains

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![Diagram showing rising chains]

$(0, 0, 3, 4, 4, 15)$

$(0, 0, 1, 1, 1, 13)$
Constructing Rising Chains

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\[(0, 0, 3, 4, 4, 15) \rightarrow (3, 3) \rightarrow (0, 0, 2, 3, 3, 15) \rightarrow (0, 0, 1, 1, 1, 13)\]
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- Let $\alpha_p = (0, 0, 3, 4, 4, 15)$ and $\alpha_q = (0, 0, 1, 1, 1, 13)$
Theorem

For every $m, n \in \mathbb{N}$, the edge-labeling $\lambda$ is an EL-labeling for $\mathcal{T}_n^{(m)}$. 
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Proposition (Björner & Wachs, 1996)

Let $P$ be a bounded poset and $[p, q]$ an interval in $P$. If $P$ is EL-shellable, then

$$\mu(p, q) = \text{number of even length falling maximal chains in } [p, q]$$
$$- \text{number of odd length falling maximal chains in } [p, q]$$
Preliminaries

EL-Shellability of $\mathcal{T}_n^{(m)}$

Topological Properties of $\mathcal{T}_n^{(m)}$

Falling Maximal Chains

Topological Consequences

**Proposition (Björner & Wachs, 1996)**

Let $\mathcal{P}$ be a bounded poset and $[p, q]$ an interval in $\mathcal{P}$. If $\mathcal{P}$ is EL-shellable, then

$$\mu(p, q) = \text{number of even length falling maximal chains in } [p, q]$$

$$- \text{number of odd length falling maximal chains in } [p, q]$$

**Theorem (Björner & Wachs, 1996)**

Let $\mathcal{P}$ be an EL-shellable poset. Then, the order complex $\Delta(\overline{\mathcal{P}})$ of $\overline{\mathcal{P}}$ has the homotopy type of a wedge of spheres, and the dimension of the $i$-th homology group of $\Delta(\overline{\mathcal{P}})$ is given by the number of falling maximal chains of length $i + 2$. 
Falling Maximal Chains

**Theorem**

Let \([p, q]\) be an interval in \(\mathcal{T}_{n}^{(m)}\). There is at most one falling maximal chain in \([p, q]\).
Falling Maximal Chains

**Theorem**

Let \([p, q]\) be an interval in \(\mathcal{T}_n^{(m)}\). There is at most one falling maximal chain in \([p, q]\).

- Let \(\alpha_p = (a_1, a_2, \ldots, a_n)\), \(\alpha_q = (b_1, b_2, \ldots, b_n)\) and let \(D = \{j \mid a_j \neq b_j \text{ and } a_j \geq a_{j-1} + m\}\) = \(\{j_1, j_2, \ldots, j_s\}\).

- If \(\alpha_{p(0)} < \alpha_{p(1)} < \cdots < \alpha_{p(s)}\) is a falling maximal chain in \([p, q]\), it must have the label sequence

\[
(j_s, a_{j_s}^{(0)}), (j_{s-1}, a_{j_{s-1}}^{(1)}), \ldots, (j_1, a_{j_1}^{(s-1)})
\]

- This follows, since each of the values \(a_{j_1}, a_{j_2}, \ldots, a_{j_s}\) must be decreased along a maximal chain in \([p, q]\) at least once.
Corollary

Let $p \leq q$ in $\mathcal{T}_n^{(m)}$. Then, $\mu(p, q) \in \{-1, 0, 1\}$. 
Conclusions

Corollary

Let $p \leq q$ in $\mathcal{T}_n^{(m)}$. Then, $\mu(p, q) \in \{-1, 0, 1\}$.

Corollary

Each open interval in $\mathcal{T}_n^{(m)}$ has the homotopy type of either a sphere or a point.
Thank you!