Limit Distributions of Continuous-State Branching Processes with Immigration

Martin Keller-Ressel
TU Berlin

based on joint work with Aleksandar Mijatovic

Dynstoch Workshop 2011, Universität Heidelberg
June 17, 2011
Part I

Introduction: Ornstein-Uhlenbeck-type processes and CBI-processes on $\mathbb{R}_{\geq 0}$
Definition (OU-type process)

$\mathbb{R}_{\geq 0}$-valued Ornstein-Uhlenbeck-type process satisfies SDE

$$dX_t = -\lambda X_t dt + dZ_t, \quad \lambda > 0,$$

$Z$ a Lévy subordinator with drift $b \in \mathbb{R}_{\geq 0}$ and Lévy measure $m(d\xi)$.

- This is the classical Ornstein-Uhlenbeck SDE with BM replaced by an increasing Lévy process.
- It follows easily that if $X_0 \in \mathbb{R}_{\geq 0}$, then $X$ is $\mathbb{R}_{\geq 0}$-valued.

Limit distributions of OU-type processes have been studied extensively.
Y has *self-decomposable distribution* if \( \forall \ c \in [0,1] \) there exists a random variable \( Y_c \), independent of \( Y \), s.t.

\[
Y \overset{d}{=} cY + Y_c.
\]

We write \( Y \in \text{SD}_+ \).

(i) Self-decomposable dist. are infinitely divisible.

(ii) Self-decomposable dist. have increased degree of regularity:

- any non-degenerate \( Y \in \text{SD}_+ \) has a density (i.e. abs. cont.);
- every distribution in \( \text{SD}_+ \) is unimodal.

**Theorem (Paul Lévy)**

*Distribution is in \( \text{SD}_+ \) \( \iff \) Lévy triplet is \( (\gamma, 0, (k(x)/x)dx) \) and\*

\[
\gamma \geq 0 \quad \text{and} \quad k : (0, \infty) \rightarrow [0, \infty) \quad \text{non-increasing.}
\]
**Theorem (Jurek and Vervaat [1983], Sato and Yamazato [1984])**

- Let $X$ be an OU-type process on $\mathbb{R}_{\geq 0}$ and suppose that $m(\,d\xi)$ satisfies the log-moment condition $\int_{\xi > 1} \log \xi \, m(\,d\xi) < \infty$. Then $X$ converges to a self-decomposable limit distribution $L$.

- Conversely, for every self-decomposable distribution $L$ on $\mathbb{R}_{\geq 0}$ there exists a unique OU-type process, such that the Lévy measure $m(\,d\xi)$ of its driver satisfies $\int_{\xi > 1} \log \xi \, m(\,d\xi) < \infty$ and $L$ is the limit distribution of the process.

One-to-one-correspondence: OU-type processes with log-moment-condition $\leftrightarrow$ self-decomposable distributions.
CBI-processes (1)

**Definition (CBI-process)**

A process $X = (X_t)_{t \geq 0}$ is a *continuous-state branching process with immigration (CBI-process)* if:

- $X$ stochastically continuous conservative Markov process on $\mathbb{R}_{\geq 0}$
- with affine Laplace exponent:
  
  $$
  - \log \mathbb{E}^X \left[ e^{-uX_t} \right] = \phi(t, u) + x\psi(t, u),
  $$

- Can be obtained as a scaling limit of Galton-Watson processes with immigration.
- The class of CBI-processes coincides with the class of affine processes on $\mathbb{R}_{\geq 0}$ in the sense of Duffie, Filipovic, and Schachermayer [2003].
- Contains all OU-type processes on $\mathbb{R}_{\geq 0}$.
Theorem (Kawazu and Watanabe [1971])

Let \((X_t)_{t \geq 0}\) be a CBI-process. Then the functions \(\phi(t, u)\) and \(\psi(t, u)\) are \(t\)-differentiable with derivatives

\[
F(u) = \left. \frac{\partial}{\partial t} \phi(t, u) \right|_{t=0}, \quad R(u) = \left. \frac{\partial}{\partial t} \psi(t, u) \right|_{t=0}
\]

which are of Levy-Khintchine form

\[
F(u) = bu - \int_{(0, \infty)} \left( e^{-u \xi} - 1 \right) m(d\xi),
\]

\[
R(u) = -\alpha u^2 + \beta u - \int_{(0, \infty)} \left( e^{-u \xi} - 1 + uh_R(\xi) \right) \mu(d\xi),
\]

where \(\alpha, b \in \mathbb{R}_{\geq 0}, \beta \in \mathbb{R}\) and \(m, \mu\) are Lévy measures on \((0, \infty)\), with \(m\) satisfying \(\int_{(0, \infty)} (x \wedge 1) m(dx) < \infty\).
CBI-processes (3)

- $F$ is the Laplace exponent of a Lévy subordinator $X^F$.
- $R$ is the Laplace exponent of a spectrally positive Lévy process $X^R$.
- One-to-one Correspondence: CBI-process $X \leftrightarrow$ Lévy processes ($X^F, X^R$).
- $\phi(t, u), \psi(t, u)$ satisfy the generalised Riccati equations:
  \[
  \begin{align*}
  \frac{\partial}{\partial t} \phi(t, u) &= F(\psi(t, u)), & \phi(0, u) = 0, \\
  \frac{\partial}{\partial t} \psi(t, u) &= R(\psi(t, u)), & \psi(0, u) = u.
  \end{align*}
  \]

- Special cases:
  (i) $X$ CB-process: $F \equiv 0$.
  (ii) $X$ OU-type process: $R(u) = \beta u$ for $\beta \in \mathbb{R}$.
  (iii) $X$ Feller diffusion: $F(u) = bu$ and $R(u) = \beta u + \frac{\alpha^2}{2} u^2$. 
Pathwise description of CBI-processes

L-K triplet of $X^R$ is $(\beta, 2\alpha, \mu(dx))$ and $W$ its BM component.

$N_1(ds, du, d\xi)$ Poisson random measure with comp. $ds du \mu(d\xi)$;
Denote the compensated Poisson random measure
$\tilde{N}_1(dt, du, d\xi) = N_1(ds, du, d\xi) - ds du \mu(d\xi)$.

Dawson and Li 2006 show that CBI-process satisfies SDE

$$dX_t = dX_t^F + \beta X_t dt + \sqrt{2\alpha X_t} \, dW_t + \int_0^{X_t-} \int_0^\infty \xi \tilde{N}_1(dt, du, d\xi)$$

under certain moment conditions on the Lévy measures $m, \mu$. 
Some applications of CBI-processes

- Natural candidates for modelling of positive, mean-reverting quantities: volatility, interest rates, default intensities.
- Diffusive behaviour and jumps can be combined.
- Processes can show self-exciting behaviour through state-dependent jump intensity.
- Tractable because the Laplace transform of the transition density is known up to a scalar ODE (gen. Riccati equation).
Part II

Results on the limit distributions of CBI-processes
Theorem (Pinsky (1972), K.-R., Mijatovic (2011))

$X$ CBI-process on $\mathbb{R}_{\geq 0}$ with $R'_+(0) < 0$. TFAE:

(a) $X$ converges to a limit distribution $L$ as $t \to \infty$;
(b) $X$ has the unique invariant distribution $L$;
(c) the measure $m(dx)$ satisfies the log-moment condition

$$\int_{\xi > 1} \log \xi \, m(dx) < \infty.$$ 

Moreover the limit $L$ has the following properties:

(i) $L$ is infinitely divisible;
(ii) the Laplace exponent $l(u) = -\log \int_{[0,\infty)} e^{-ux} \, dL(x)$ of $L$ is given by

$$l(u) = -\int_{0}^{u} \frac{F(s)}{R(s)} \, ds \quad (u \geq 0).$$
Theorem (Pinsky (1972), K.-R., Mijatovic (2011))

\( X \) CBI-process on \( \mathbb{R}_{\geq 0} \) with \( R'_+(0) < 0 \). TFAE:

(a) \( X \) converges to a limit distribution \( L \) as \( t \to \infty \);
(b) \( X \) has the unique invariant distribution \( L \);
(c) the measure \( m(\mathrm{d}\xi) \) satisfies the log-moment condition

\[
\int_{\xi > 1} \log \xi \; m(\mathrm{d}\xi) < \infty.
\]

Moreover the limit \( L \) has the following properties:

(i) \( L \) is infinitely divisible;
(ii) the Laplace exponent \( l(u) = - \log \int_{[0,\infty)} e^{-ux} \mathrm{d}L(x) \) of \( L \) is given by

\[
l(u) = - \int_0^u \frac{F(s)}{R(s)} \mathrm{d}s \quad (u \geq 0).
\]
What is the Lévy-Khintchine triplet for $L$?

### Definition (Scale Function)

The *scale function* of the dual $\hat{X}^R = -X^R$, which is spectrally negative, is the unique increasing continuous function $W : (0, \infty) \to [0, \infty)$ that satisfies

$$\int_0^\infty e^{-ux} W(x) \, dx = -\frac{1}{R(u)} \quad \text{for all } u > 0.$$

- Under the assumption that $R'_+(0) < 0$, we have

$$W(x) = -\frac{1}{R'_+(0)} \mathbb{P}\left[ \sup_{t \geq 0} X^R_t \leq x \right] \quad \text{for } x > 0.$$
Some properties of the scale function $W$ of the dual $X^R$:

- $W$ has non-degenerate right limit at zero:

$$\lim_{x \downarrow 0} W(x) = W_+(0) = \mathbb{P}\left[ \sup_{t \geq 0} X^R_t = 0 \right].$$

Excursion theory for $X^R$ implies further:

- $W$ is log-concave and hence continuous on $[0, \infty)$;
- the scale function has left- and right-derivative on $(0, \infty)$;
- if $\alpha > 0$ (recall $(\beta, \alpha, \mu(dx))$ is the L-K triplet of $X^R$), then $W \in C^2(0, \infty)$. 

Theorem (K.-R., Mijatovic (2011))

If CBI-process $X$ converges to the limit dist. $L$, then $L$ is infinitely divisible with Laplace exponent

$$l(u) = u\gamma - \int_{(0,\infty)} (e^{-xu} - 1) \frac{k(x)}{x} \, dx,$$

where $\gamma \geq 0$ and $k : (0, \infty) \to \mathbb{R}_{\geq 0}$ are given by

$$\gamma = b W(0),$$

$$k(x) = b W'(x) + W(x)m(x, \infty) + \int_{(0,x]} [W(x) - W(x - \xi)] \, m(d\xi).$$
For an OU-type process $\hat{X}^R = \lambda t$, $W(x) = \frac{1}{\lambda} 1_{[0, \infty)}(x)$ and we re-obtain $k(x) = \frac{1}{\lambda} m(x, \infty)$.

Last integral is always finite since $W$ is left-differentiable at $x > 0$.

Analogous representations of function $k(x)$, based on the law of the ultimate supremum $\sup_{t \geq 0} X^R_t$ or on the Itô excursion measure for the Poisson point process of excursions from supremum of $\hat{X}^R$, are also available.

Formula for $k(x)$ looks like a Lévy generator applied to the scale function $W(x)$. Rigorous result possible?
Second representation formula: \( k = -A_{X^F} W \)

Pick \( h : (0, \infty) \to (0, \infty) \) such that
- \( h \) is continuous and bounded;
- \( \lim_{x \downarrow 0} h(x) = 0; \)
- \( h(x) \sim e^{-cx} \) as \( x \to \infty \) for some \( c > 0 \).

Define the weighted Banach space

\[
L^h_{1}(0, \infty) := \left\{ f \in L^1_{\text{loc}}(0, \infty) : \int_0^\infty |f(x)| h(x) \, dx < \infty \right\}.
\]

Semigroup of \( X^F \) acts on \( L^h_{1}(0, \infty) \) and \( W \) is in the domain of its generator \( A_F \). Theorem implies that the following holds

\[
k = -A_{X^F} W.
\]
If $X$ is $\mathbb{R}_{\geq 0}$-valued OU-type processes, then $L$ is self-decomposable.

Sato 1999 shows that in this case we have:

(i) the support of $L$ is $[b/\lambda, \infty)$;
(ii) the distribution of $L$ is absolutely continuous;
(iii) the asymptotic behaviour of the density of $L$ at $b/\lambda$ is determined by $c = \lim_{x \downarrow 0} k(x)$.

What are the properties of a general limit $L$ of a CBI-process?
Properties of the limit distribution $L$: the degenerate case

**Definition**

A CBI-process $X$ is *degenerate of the* 

(i) *first kind*, if it is deterministic for all $X_0 = x \in \mathbb{R}_{\geq 0}$;  

(ii) *second kind*, if it is deterministic when started at $X_0 = 0$.

**Proposition (K.-R., Mijatovic (2011))**

Let $X$ a CBI-process with limit distribution $L$.

- $X$ deg. of the first kind $\implies$ $\text{supp } L = \{-b/\beta\}$;  
- $X$ deg. of the second (not first) kind $\implies$ $\text{supp } L = \{0\}$.
Support of \( L \): the non-degenerate case

The effective drift \( \lambda_0 > 0 \) of \( \hat{X}^R \) is defined as

\[
\lambda_0 = \begin{cases} 
\int_{(0, \infty)} h_R(\xi) \mu(\mathrm{d}\xi) - \beta, & \text{if } X^R \text{ has bounded variation,} \\
+\infty, & \text{if } X^R \text{ has unbounded variation,}
\end{cases}
\]

where \( h_R(\xi) = \xi / (1 + \xi^2) \) is truncation function in \( \mathbb{R} \).

**Proposition (K.-R., Mijatovic (2011))**

\( X \) a non-degenerate CBI-process with limit distribution \( L \). Then

\[
\text{supp } L = [b/\lambda_0, \infty),
\]

and hence

\[
\text{supp } L = \mathbb{R}_{\geq 0} \iff b = 0 \text{ or paths of } X^R \text{ have inf. variation.}
\]
Absolute continuity of the limit distribution \( L \)

**Proposition (K.-R., Mijatovic (2011))**

\( X \) CBI-process with limit distrib. \( L \) and \( \lambda_0 \) effective drift of \( X^R \). Then the following holds:

- **L is absolutely continuous on** \( \mathbb{R}_{\geq 0} \) **if and only if**
  \[
  \int_0^1 \frac{k(x)}{x} \, dx = \infty;
  \]

- **L is absolutely continuous on** \( \mathbb{R}_{\geq 0} \setminus \{ b/\lambda_0 \} \) **with an atom at** \( \{ b/\lambda_0 \} \) **if and only if**
  \[
  \int_0^1 \frac{k(x)}{x} \, dx < \infty.
  \]
\( \text{SD}_+ \subsetneq \text{CLIM} \subsetneq \text{ID}_+ \)

- ID\(_+\) infinitely divisible distributions on \( \mathbb{R}_{\geq 0} \);
- SD\(_+\) self-decomposable distributions on \( \mathbb{R}_{\geq 0} \).

**Definition**

\text{CLIM}: limit distribs on \( \mathbb{R}_{\geq 0} \) of CBI-processes with \( R'_+(0) < 0 \).

**Proposition (K.-R., Mijatovic (2011))**

\[ \text{SD}_+ \subsetneq \text{CLIM} \subsetneq \text{ID}_+ \]
Proof.

- all distributions in $SD_+$ are either degenerate or absolutely continuous;
- all distributions in $CLIM$ are absolutely continuous on $\mathbb{R}_{\geq 0} \setminus \{b/\lambda_0\}$, but some concentrate non-zero mass at $\{b/\lambda_0\}$;
- the class $ID_+$ contains singular distributions.

There are also concrete examples of non-selfdecomposable limit distributions of CBI-processes...
Example: non-selfdecomposable limit distribution

- $X^F$: generating triplet $(b, 0, m(dx))$;
- $X^R$: characteristic exponent $R(u) = -\alpha u^2 + \beta u$ (and $\beta < 0$).

**Scale function of dual process $\hat{X}^R$:**

$$W(x) = \left[ \exp \left( \frac{x\beta}{\alpha} \right) - 1 \right] / \beta \quad \text{and} \quad W'(x) = \exp \left( \frac{x\beta}{\alpha} \right) / \alpha.$$ 

**$k$-function of limit distribution $L$:**

$$k(x) = e^{x\beta/\alpha} \left[ \frac{b}{\alpha} + \frac{1}{\beta} \left( m(x, \infty) + \int_{(0,x]} \left( 1 - e^{-\xi\beta/\alpha} \right) m(d\xi) \right) \right] - \frac{m(x, \infty)}{\beta}$$

Choose $X^F$ as compound Poisson pr. with exp. jumps

$m(x, \infty) = e^{-x}$, $b = 0$, $\alpha = 1/2$, $\beta = -1 \implies k(x) = 2(e^{-x} - e^{-2x})$.

$\Rightarrow k$ non-mononote $\Rightarrow L$ not self-decomposable.
Sufficient conditions for $L$ to be self-decomposable

- $X^F$: generating triplet $(b, 0, m(dx))$;
- $X^R$: generating triplet $(\beta, 2\alpha, \mu(dx))$.

**Proposition**

$L$ limit distribution of CBI-process $X$. $L$ self-decomposable if:

(a) $\mu = 0$ and $\alpha = 0$,
(b) $\mu = 0$ and $m = 0$,
(c) $m = 0$ and $W$ is concave on $(0, \infty)$.

Conversely, if $m = 0$ and $L$ is self-decomposable, then $W$ is concave on $(0, \infty)$. 
Open problems

(i) Characterise limit of CBI-process $X$ in the boundary case that $R'_+(0) = 0$.

(ii) Find structural characterisation of the possible limit distributions using formula

$$k(x) = b\, W'_+(x) + W(x)m(x, \infty) + \int_{(0,x]} [W(x) - W(x - \xi)] \, m(d\xi).$$

- for OU-type processes the limit class has a structural description as self-decomposable distributions.
Thank you for your attention!

Preprint:


